

# Engineering Systems for Allocating Public Goods

Lecture 2: Truthfulness and the Core

# Recall: Three Important Goals

- Efficiency            Last Class
- Truthfulness        Today
- Fairness             Next Class

# Plan for the Day

1. Revisiting Pareto Efficiency
  - Definition (review)
  - Finding all Pareto efficient allocations
  - Other ways to find Pareto efficient allocations
2. Introduction to incentives
  - Definition: Truthfulness
  - Group work
3. Allocation with pre-existing ownership
  - Application: Board Game “Math Trades”
  - Key concepts: individually rational, the core

# Definition of Pareto Efficiency

An allocation is Pareto efficient if it is impossible to make somebody better off without making somebody else worse off.

1	2	3	4
A	B	D	B
B	C	A	D
C	D	C	A
D	A	B	C



This allocation cannot be improved by two-person trades, but is NOT Pareto efficient.

(All agree that ACDB is better.)

# Homework Review

1. Only one Pareto efficient allocation?  
If (and only if) it is possible to give everyone their first choice.
2. Every feasible allocation is Pareto efficient?  
If (and only if) everyone has the same preferences.

Intuition:

- When people have different preferences, not many PE allocations.
- When people have similar preferences, many PE allocations.

Does Serial Dictatorship find **EVERY**  
Pareto Efficient Allocation?

# Dynamic

vs

# Direct

- ✦ Doesn't require participants to provide as much information.
- ✦ Reasonable if either number of people or number of prizes is small.
- ✦ Can be good if people have complex preferences (i.e. over teammates).

- ✦ Only requires one round of back and forth, and thus may take less time.
- ✦ Reasonable if not too many options for people to rank.

**Discussion:** Would you recommend a direct or dynamic implementation?

SPPS: 2700 students assigned to 11 high schools.

CBS Offices: 20 faculty assigned to 20 offices.

IE 5541: 50 students assigned to 10 project teams.

# Other Ways To Find PE Allocations

## “First Choices First” Algorithm

1. Give as many agents their first choice as possible.
2. Give as many of the remaining agents their second choice as possible.
3. Give as many of the remaining agents their third choice as possible.

⋮

To make this well-defined, suppose we break ties in favor of low-numbered agents.

## Optimization Algorithms

Minimize the sum of ranks, breaking ties in favor of low-numbered agents.



# Lecture 1 Study Guide

## Concepts

- Allocation (Assignment)
- Preference Profile
- Pareto Efficient (PE)
- Rank Efficient
- Dynamic Mechanism
- Direct (Revelation) Mechanism

## Algorithms

- Serial Dictatorship
- First Choices First (Boston)

## Facts

- Serial Dictatorship is PE
- First Choices First is PE.
- Neither is guaranteed to be Rank Efficient.

# Three Ways to Find Pareto Efficient Allocation

1. Serial dictatorship.
2. First choices first.
3. Rank efficiency (optimization).

Should we always use optimization?

How do we know what preferences to use? **Agents tell us!**

Should they tell the truth?

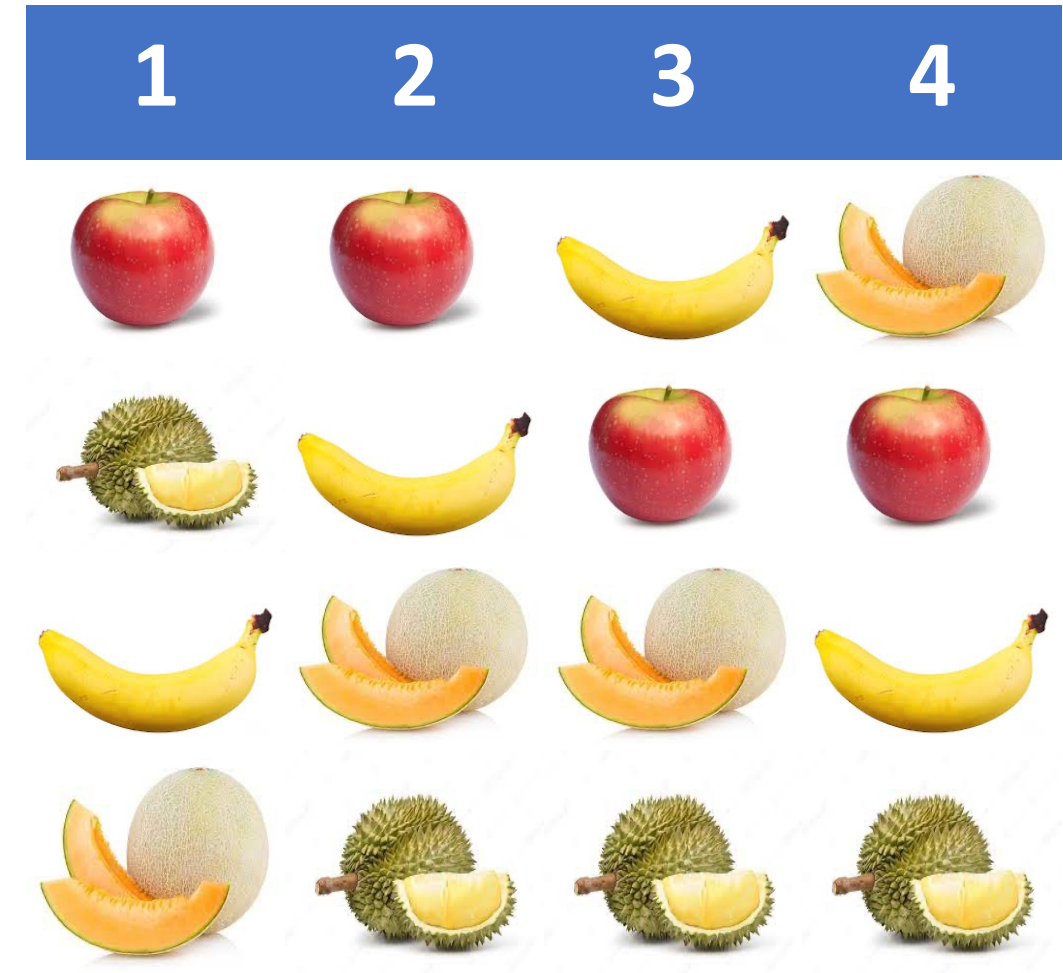
# Group Work

Consider three possible algorithms:

1. Find a rank-efficient allocation. Break ties in favor of low-numbered agents.
2. Use first-choices-first. Break ties in favor of low-numbered agents.
3. Use serial dictatorship. Let low-numbered agents choose first.

In each case,

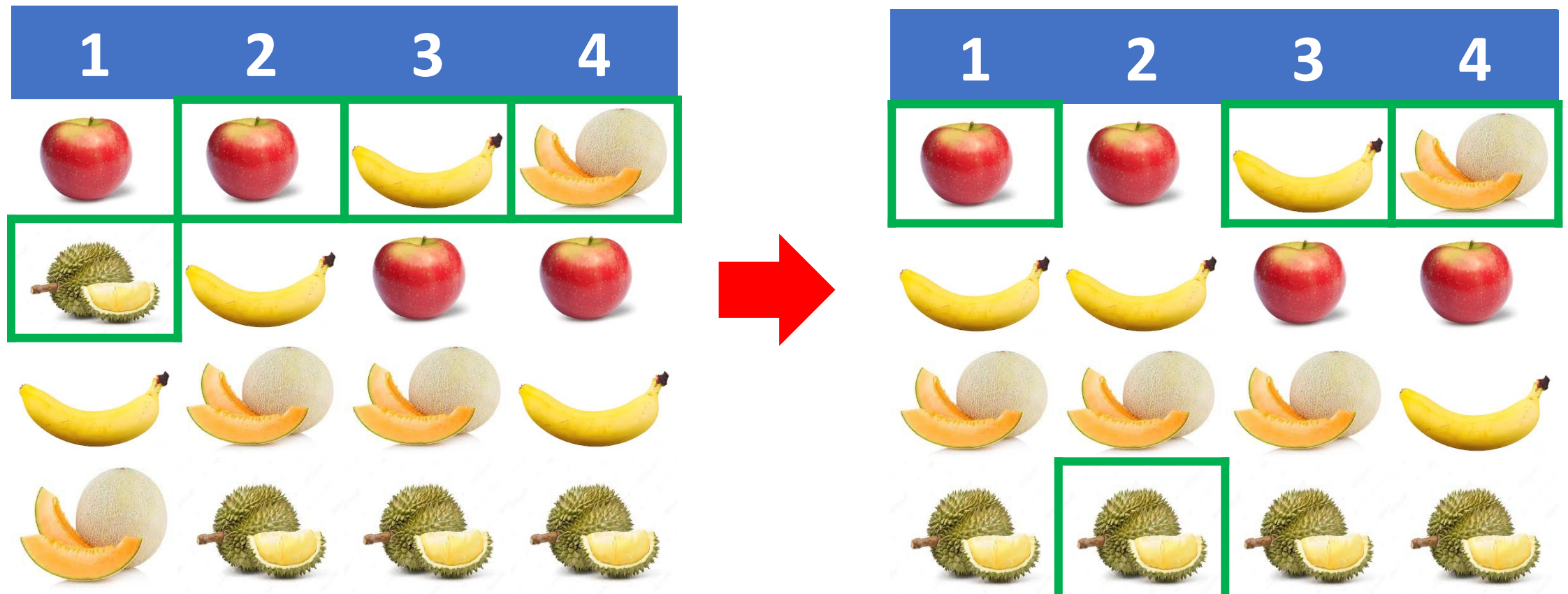
- a) What allocation results?
- b) Could any agent improve their outcome by reporting different preferences?



# Analysis: Rank Efficient Allocation

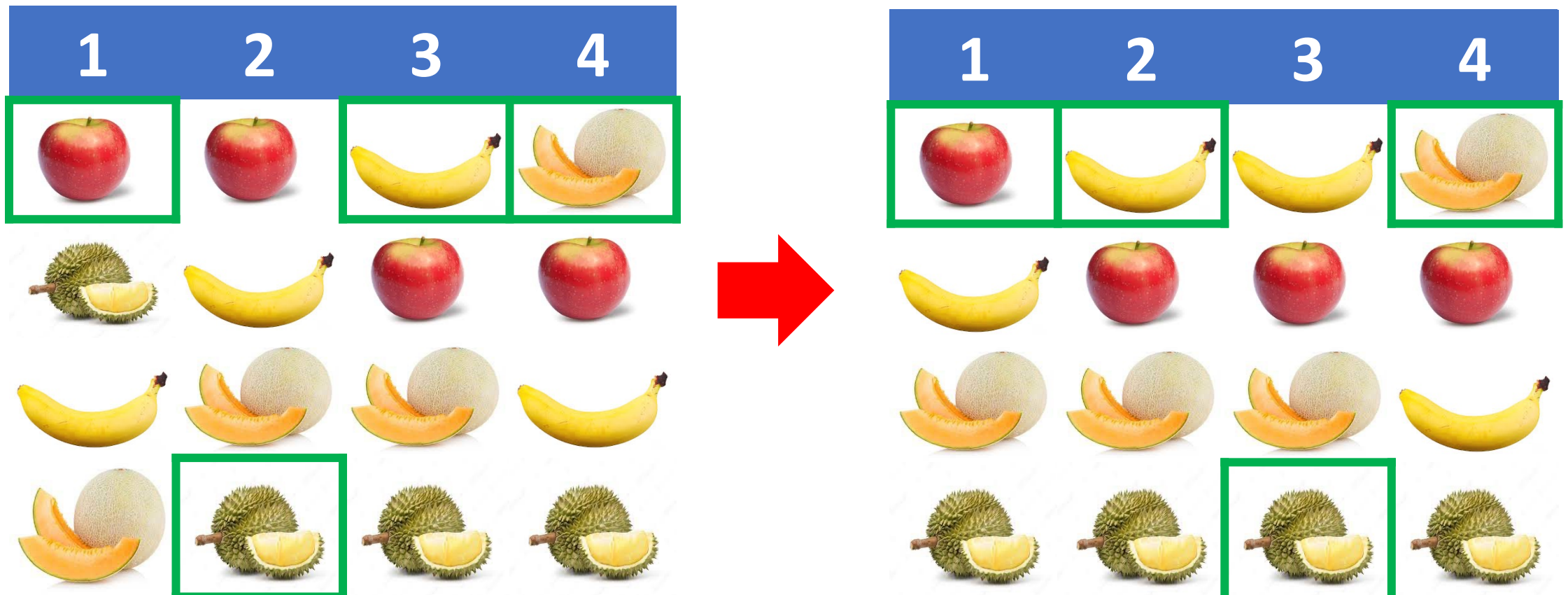
We gave the apple to Agent 2 because Agent 1 likes durian the most.

What if Agent 1 lies, and says they hate durian?



















# Analysis: First Choices First

We gave the banana to Agent 3 because Agent 2 ranked it second.  
What if Agent 2 lies, and says bananas are their favorite?



# Analysis: Serial Dictatorship

Changing your report does not change choices of agents before you.  
You are already getting your favorite item that remains!

1	2	3	4
			
			
			
			

# Incentivizing Truthful Reporting

A **mechanism** is a function from preference profiles to allocations.

A mechanism is **truthful** (incentive compatible, strategy-proof) if nobody can ever benefit from lying:

for *every* agent, *every* possible preference of that agent, and *every* possible report by others, the agent does best by reporting truthfully.

Break



# “Math Trades” For Board Games

# Background: Math Trades

Today, we will assume:

- i. Each person brings only one game.
- ii. No two people bring the same game.
- iii. Everybody ranks all the games.

Just like the fruit example, but now everyone owns something to start!

How many Pareto efficient allocations can you find in this example?



# There are Three Pareto Efficient Allocations

## ABC



## CBA



## ACB



## Group Work:

1. Which allocation would you recommend, and why?
2. Make a case against your choice in Part 1.

# What if we suggest allocation ACB?

ABC

CBA

ACB



A mechanism is **individually rational** if agents are never harmed by participating.



# Used in Practice: TradeMaximizer

1. Find an individually rational allocation.
2. Subject to this constraint, maximize total number of games traded.
3. Subject to this constraint, minimize sum of ranks.
4. Further tiebreakers.

## Group Work:

1. Which allocation would TradeMaximizer select?
2. Is TradeMaximizer truthful?



# What if we suggest allocation CBA?

ABC



CBA



ACB



Agent 1 can refuse to trade Bohnanza unless her or she gets Azul in return.  
Agent 1 can call Agent 2 and arrange to swap (ignoring our suggestion).

# Will People Follow Your Suggestion?

A proposed allocation is **individually rational (IR)** if no agents are harmed by participating.

A proposed allocation is in the **core** if no group of agents can collectively deviate to an allocation that is at least as good for everyone in the group and better for someone in the group.

Informal Meaning:

- Individually Rational: no *individual* can benefit from ignoring our suggestion.
- Pareto Efficient: the *entire group* cannot benefit from ignoring our suggestion.
- Core: no *subset of agents* can benefit from ignoring our suggestion.

Therefore, all core allocations are individually rational + Pareto efficient.

All Allocations

Pareto  
Efficient

Core

Individually  
Rational

ABC ACB  
BCA BAC  
CAB CBA

**Group Work:**

For the board game example, there are 6 possible allocations. Place each allocation in its appropriate location above.





# All Allocations

Pareto Efficient

ACB

Core  
ABC

CBA

BAC CAB

Individually Rational

BCA



## Group Work:

For the board game example, there are 6 possible allocations. Place each allocation in its appropriate location above.

# Several Natural Questions

1. Is there always an allocation in the core?
2. How can we find an allocation in the core?
3. Can there be more than one allocation in the core?

# Top Trading Cycles

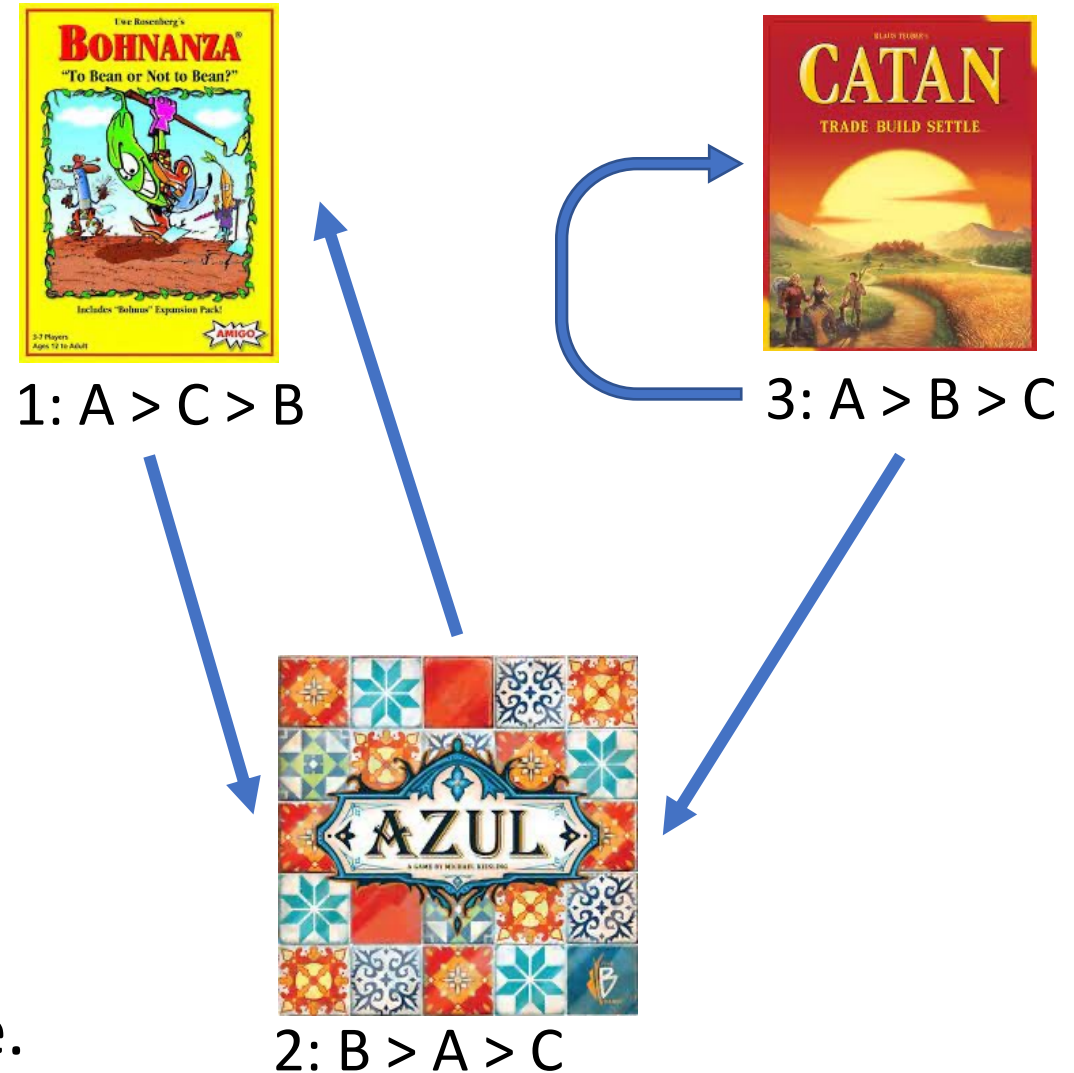
**Observation:** if a group can trade with each other and all get their favorite game, then if you recommend anything else, they will ignore you.

## Algorithm:

















1. Each person “points” to their favorite remaining game.
2. Everyone that is part of a “loop” gets the game they are pointing to.
3. Return to Step 1.

**Note:** People can point to their own game.

**Question:** Is there always a loop? ✓



# Top Trading Cycles Practice

1	2	3	4
			
			
			
			

## Group Work:

Suppose we start from the allocation shown, and apply Top Trading Cycles.

1. What cycle clears first?
2. What is our final allocation?

# Top Trading Cycles Practice

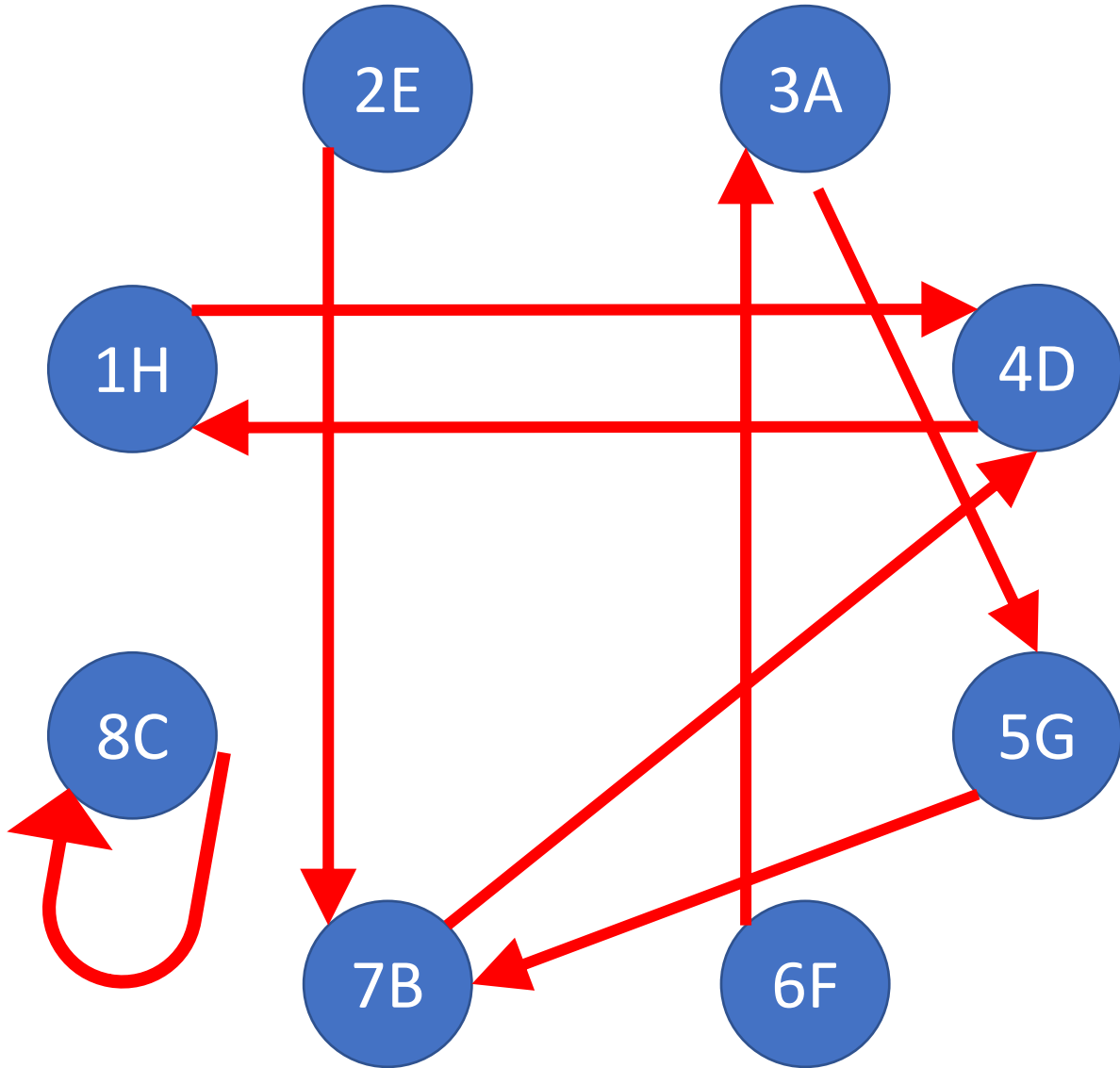
1	2	3	4	5	6	7	8
D	B	G	H	B	A	D	C
G	D	F	D	C	E	F	E
B	C	A	E	G	G	A	D
E	F	C	A	E	H	G	G
F	G	E	G	F	C	B	B
C	E	H	B	A	F	C	F
H	H	D	C	H	B	H	A
A	A	B	F	D	D	E	H

## Group Work:

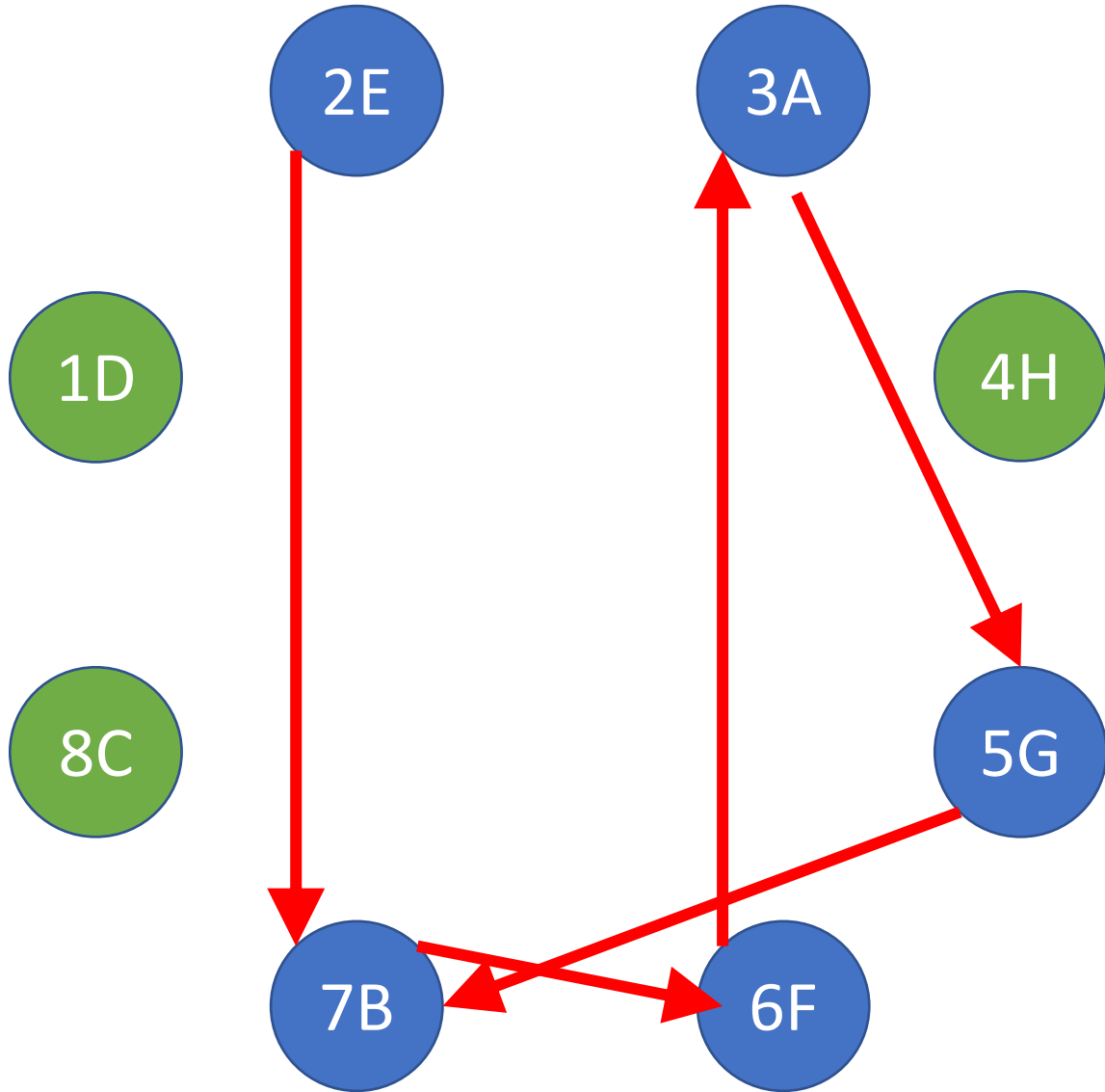
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1. What cycle clears first?
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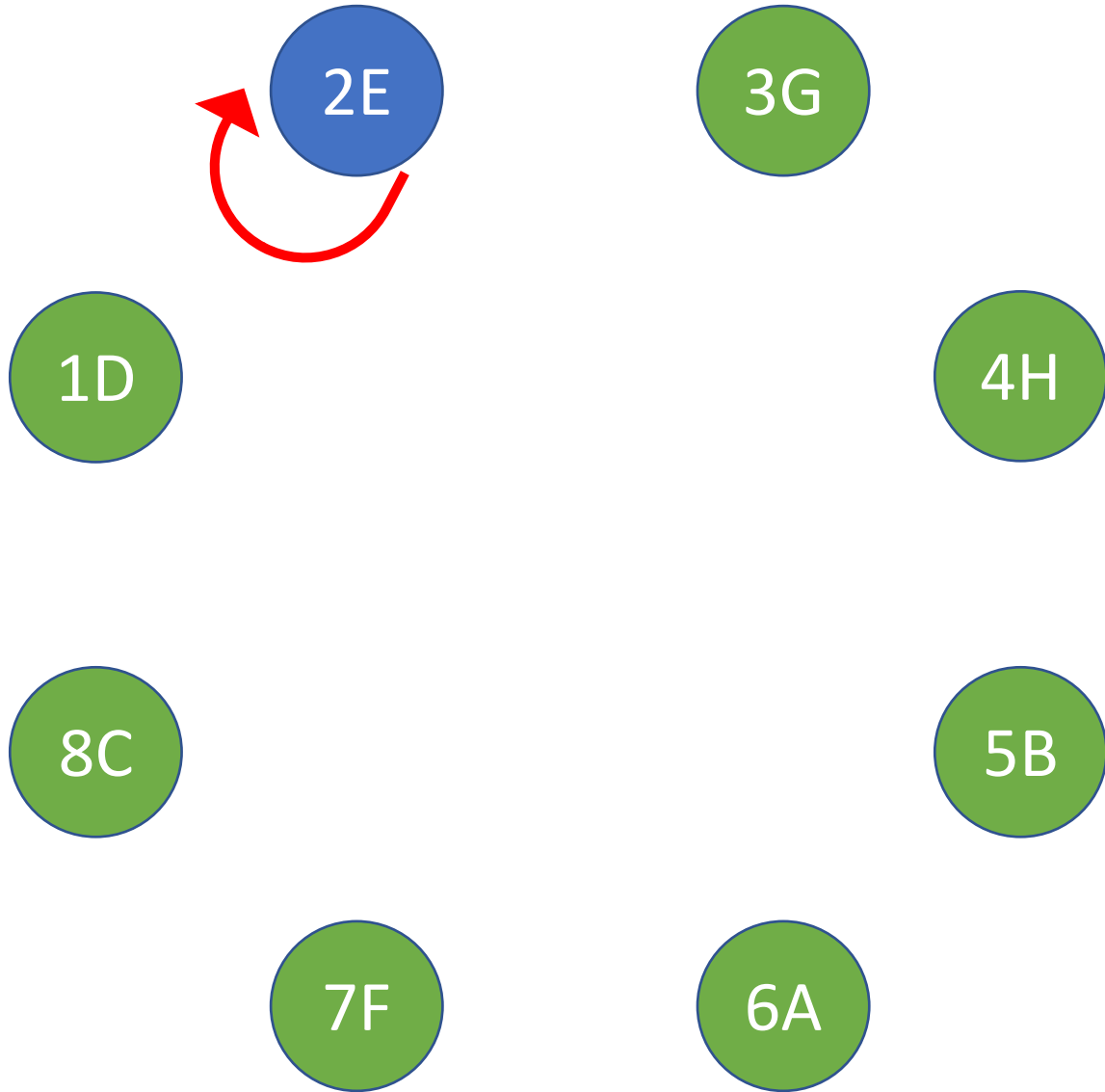
# Initial Pointing



After removing initial cycles



After two rounds of removing cycles





# Final Allocation

1	2	3	4	5	6	7	8
D	B	G	H	B	A	D	C
G	D	F	D	C	E	F	E
B	C	A	E	G	G	A	D
E	F	C	A	E	H	G	G
F	G	E	G	F	C	B	B
C	E	H	B	A	F	C	F
H	H	D	C	H	B	H	A
A	A	B	F	D	D	E	H

# Top Trading Cycles Incentives

1	2	3	4	5	6	7	8
D	B	G	H	B	A	D	C
G	D	F	D	C	E	F	E
B	C	A	E	G	G	A	D
E	F	C	A	E	H	G	G
F	G	E	G	F	C	B	B
C	E	H	B	A	F	C	F
H	H	D	C	H	B	H	A
A	A	B	F	D	D	E	H

## Group Work:

Reset to this initial state.

Repeat top trading cycles, but now you can point to anyone.

Try to get a better object than last time. Can you?

# Nice Features of Top Trading Cycles

1. It produces the unique allocation in the core.  
( $\Rightarrow$  Pareto efficient, Individually rational)
2. It is strategy-proof!
3. It is the **only** mechanism that is Pareto efficient, individually rational, and strategy-proof. (*Ma 1994: Strategy-Proofness and the Strict Core in a Market with Indivisibilities*)

# Summary

We want people to participate, tell us their true preferences, and follow our recommendations. (Otherwise, even a “good” recommendation is useless!)

**Individually Rational** (allocation): Nobody is harmed by participating.

**Truthful** (mechanism): Nobody benefits from lying.

**Core** (allocation): No group can benefit by deviating from recommendation.

There is only one core allocation, and it is found by Top Trading Cycles.

Amazingly, there is only one mechanism that is truthful and individually rational: recommend the unique core allocation!

# Coming Up

1. Can we make Serial Dictatorship and Top Trading Cycles “fair”?
2. How to define fairness?
3. What other mechanisms are Pareto efficient, truthful, and fair?

# Study Guide

## Concepts

- Endowment
- Truthful

(Strategy-Proof,  
Incentive Compatible)

- Individually Rational
- The Core

## Algorithms

- Top Trading Cycles

## Facts

- Top Trading Cycles gives the unique allocation in the core
- Top Trading Cycles is PE, SP, IR.
- Top Trading Cycles is the **only** algorithm with these properties.