

Fair and Efficient Discrete Resource Allocation: A Market Approach*

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Abstract

In a variety of cases, a set of indivisible objects must be allocated to a set of agents where each agent needs exactly one object. Examples include the allocation of tasks to workers, spots at public schools to pupils, and kidneys to patients with renal failure. We consider the mixed ownership case of this problem (some objects are initially owned by some agents while the other objects are unowned) and introduce a market-based mechanism that is procedurally reminiscent of the Walrasian Mechanism from equal-division. Our mechanism is strategy-proof and procedurally fair, and it leads to Pareto-efficient allocations. We obtain that it is equivalent to a well-known priority-order based mechanism. The equivalence result in the classical paper by Abdulkadiroğlu and Sönmez (*Econometrica*' 1998) follows as a corollary.

Key Words: Resource Allocation; Indivisible Objects; Matching; House allocation; House allocation with existing tenants; Housing lottery

JEL Codes: C71; C78; D71; D78

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1 Introduction

We consider the problem of allocating n indivisible objects to n agents where each agent needs exactly one object and agents' preferences over objects are strict. Monetary transfers are not allowed.¹ There are numerous real-life applications of this problem, such as the allocation of tasks to workers, spots at public schools to pupils, kidneys to patients with renal failure, dormitory rooms to college students, and legislators to committees [1, 2, 3, 5, 9, 15]. The purpose of this paper is to design a mechanism (a systematic allocation rule) which has a market-based approach, is fair in a certain sense, and leads to efficient (Pareto optimal) allocations (matchings of agents and objects). A core issue to pay attention to when designing a mechanism is that preferences are elicited from agents who may respond strategically rather than truthfully.

As is the convention in this literature, we employ the paradigm of allocating “houses” to agents. There are three cases of this problem in the literature, varying in the initial ownership structure. In the private ownership case, called a *housing market*, initially each agent (privately) owns a house (see Shapley and Scarf [18]). In the social ownership case, called a *house allocation problem*, initially houses are unowned (meaning that they are socially owned by all agents; see Hylland and Zeckhauser [9]). In the mixed ownership case, called a *house allocation problem with existing tenants*, the previous two cases are generalized: There are k “newcomers” who initially do not own any houses; k “vacant houses,” which are initially unowned; and $n - k$ “existing tenants” and $n - k$ “occupied houses” such that each existing tenant owns her occupied house (see Abdulkadiroğlu and Sönmez [2]).

This paper considers the mixed ownership case. Besides being more general, it draws attention because of a number of interesting real-life applications of it. A prominent example is *kidney exchange with Good Samaritan Donors*. In the most serious forms of renal disease, the preferred treatment is kidney transplantation. As of March 2009, there were about 79,000 patients waiting for a kidney transplant in the United States. While some patients (“existing tenants”) have friends or relatives willing to donate them their kidneys (“occupied

¹We do not take a normative standpoint against the use of money. In many real-life applications, however, the use of money is not permissible.

houses”), there are also patients (“newcomers”) who do not have donors. There are also kidneys obtained from Good Samaritan Donors and cadavers (“vacant houses”) which are donated to patients collectively. In many cases a patient cannot be transplanted the kidney of her donor due to medical incompatibilities. A common practice is then kidney exchange in which patients are transplanted the kidneys of one another’s donors. A kidney exchange may also involve patients without donors and kidneys obtained from Good Samaritan Donors and cadavers [15, 17, 22]. Another real-life application of the mixed ownership case is *allocation of on-campus housing units* at college campuses . Each returning student (“existing tenant”) occupies a room from the previous year. There are also incoming freshmen (“newcomers”), who initially do not occupy any rooms, and vacant rooms, vacated by the graduated class [2, 7].

The mechanism that we will introduce is inspired from the “Walrasian Mechanism from equal-division,” which is arguably the most widely advocated mechanism to allocate a socially owned bundle of (infinitely) divisible goods to a set of agents fairly and efficiently. This mechanism proceeds in a simple manner. First, the bundle is equally divided among agents. If agents also have private endowments, the natural idea is to add agents’ equal-division shares of the bundle to their private endowments. This results in a private ownership exchange market, in which the Walrasian Mechanism is operated, producing a Walrasian allocation, which is Pareto-efficient under standard assumptions on preferences.² This mechanism has been shown to be compatible with many equity criteria that has been proposed in the literature (see Thomson [21]).

Clearly, “indivisible” objects cannot be equally “divided.” A *probabilistic* equal-division idea can still be employed, however, using random distribution. In an indivisible object allocation problem, the first mechanism that involves random distribution has been introduced by Abdulkadiroğlu and Sönmez [1]. In the context of a house allocation problem, they proposed the *core from random endowments* mechanism (in short, CFRE), which proceeds as follows:

Distribute n houses to n agents uniformly at random such that each agent receives

²A Walrasian allocation is an allocation that can be attained in a Walrasian equilibrium. The Walrasian mechanism is the rule that maps a private ownership exchange economy to a Walrasian allocation.

exactly one house. In the induced exchange market, reallocate houses to agents by executing the *top trading cycles* mechanism³ (in short, TTC), as described below:

Step 1: Let each agent “point” to her most preferred house and let each house “point” to its owner. There exists at least one “cycle,” characterized by a list a^1, a^2, \dots, a^j of agents where a^1 points to a^2 's house which points to a^2 ; a^2 points to a^3 's house which points to a^3 ; \dots ; a^{j-1} points to a^j 's house which points to a^j ; and a^j points to a^1 's house which points to a^1 .⁴ Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

Roth [13] showed that TTC is strategy-proof (truthful preference revelation is a dominant strategy), from which it immediately follows that CFRE is also strategy-proof.

There are interesting procedural and theoretical parallels between the Walrasian Mechanism from equal-division and CFRE. For both mechanisms, first a private ownership exchange market is generated by distributing the socially owned resources to agents employing an equal-division idea. In the former, equal-division is in the physical sense; in the latter, in a probabilistic sense, as each socially owned vacant house is given to each agent with exactly the same probability, $1/n$. For both mechanisms, market outcomes are calculated in the induced exchange markets. In the former, the Walrasian Mechanism is operated producing a Walrasian allocation, which is Pareto-efficient. In the latter, TTC is executed, which is the exact counterpart of the Walrasian Mechanism in this context, as it produces the unique Walrasian allocation, which is again Pareto-efficient (see Shapley and Scarf [18]). In the former, under certain assumptions that guarantee core convergence (see Aumann [4]) and the uniqueness of a Walrasian allocation (see Mas-Colell [11]), the Walrasian Mechanism produces the exchange market's unique core allocation. In the latter, similarly, TTC produces

³This mechanism is credited to David Gale in Shapley and Scarf [18].

⁴The existence of a cycle arises from the fact that the number of agents and houses is finite.

the exchange market's unique core allocation (see Shapley and Scarf [18]).

We want to employ a similar approach in the mixed ownership case. The random distribution in the mixed ownership context is a more delicate issue, however. To highlight the key challenges, consider the following two mechanisms, the first of which is due to Sönmez and Ünver [19]:

- (1) Distribute k vacant houses to k newcomers uniformly at random such that each newcomer receives exactly one vacant house. Then reallocate houses to agents by executing TTC.
- (2) Distribute k vacant houses to n agents uniformly at random such that out of n agents k of them receive a vacant house. Then reallocate houses to agents by executing TTC.

In (1), random distribution results in a housing market and TTC produces its unique core allocation. There is no probabilistic equal-division, however, as a vacant house, which is socially owned, is given to a newcomer with $1/k$ probability but to an existing tenant with zero probability. To emphasize the fairness shortcoming of this mechanism, consider an existing tenant whose occupied house is the least desired house of every agent. Then in (1) she will be assigned her least desired house. In a sense, she is punished for owning a house, which is not what we desire. Arguably, this feature of the mechanism may also cause an incentive shortcoming. The agent who owns the least desired house may respond strategically by first giving up her house and then participating in the mechanism as a newcomer.

In (2), there is a probabilistic equal-division, as each vacant house is given to each agent with exactly $1/n$ probability. There is an efficiency shortcoming of this mechanism, however. After vacant houses have been distributed, if an existing tenant e receives a vacant house and a newcomer a does not, then, in the induced exchange market, e owns two houses (a vacant house besides her occupied house) while a owns none. When TTC is executed, a cannot join a cycle and remains unassigned, and when e departs from the market by joining a cycle, one of her two houses remains, which becomes “wasted.”

We introduce a mechanism that resolves the fairness and efficiency tension in (1) and (2)

in an intuitive way. The *core from random distribution* mechanism (in short, CFRD) proceeds as follows:

Besides the k vacant houses, introduce $n - k$ “inheritance rights” associated with $n - k$ existing tenants. Distribute k vacant houses and $n - k$ inheritance rights to n agents uniformly at random such that each agent receives exactly one vacant house or one inheritance right. If an agent receives an inheritance right, she becomes the “inheritor” of the associated existing tenant; if she receives a vacant house, she then owns that vacant house. Therefore, in the induced exchange market, a newcomer owns a house or an inheritance right, and an existing tenant owns two houses (a vacant house besides her occupied house) or an inheritance right and a house. We call this an *inheritors augmented housing market*. In this market reallocate houses to agents by executing the following *inheritors augmented top trading cycles* mechanism (in short, IATTC):

Step 1: Let each agent point to her most preferred house and let each house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

The innovation in CFRD is the role played by inheritance rights in the execution of IATTC. When an existing tenant who owns two houses is removed after joining a cycle, her remaining house is not wasted; rather, it is given to her inheritor (or to the inheritor of her inheritor

and so on). Note that this has no negative welfare implications for her. She can trade any of her two houses to join a cycle and be assigned the best house that she can. Her remaining house is given to another agent only after she departs from the market. It can be shown that IATTC is strategy-proof.⁵ Then it immediately follows that CFRD is also strategy-proof.

Observe that there is a probabilistic equal-division in CFRD; each vacant house is given to each agent with exactly $1/n$ probability. Further, IATTC is the counterpart of the Walrasian Mechanism in this context, as it produces a Walrasian allocation.⁶ In Theorem 1 we show that, as TTC produces the unique core allocation in a housing market, IATTC produces the unique core allocation in an inheritors augmented housing market. The core allocation notion in this context is more subtle, however. The definition should incorporate the rights of inheritors (see Definition 2). Also, observe that in an inheritors augmented housing market where each agent owns exactly one house, IATTC proceeds just like TTC; in this sense, an inheritors augmented housing market and IATTC can be seen as generalizations of a housing market and TTC.

Our main theoretical contribution is a surprising equivalence result, similar to the one in Abdulkadiroğlu and Sönmez [1]. In a house allocation problem, they showed that CFRE is equivalent to *random-priority* (also known as “random serial dictatorship”),⁷ which proceeds as follows: (If two mechanisms are equivalent, it means, given any preference profile, any given allocation is produced by the two mechanisms with exactly the same probability.)

Choose a priority-order (an ordering of agents) uniformly at random. Then execute the associated *priority-rule* as follows: Assign the first agent her most preferred house, the second agent her most preferred house among remaining ones, and so on.

In Theorem 2 we show that CFRD is equivalent to *random You request my house - I get your turn* (in short, *random Y-I*), a mechanism due to Abdulkadiroğlu and Sönmez [2] and which proceeds as follows:

⁵We do not include a proof for this, as it follows as a corollary of Theorem 2.

⁶This is due to Theorem 2 and the fact that a Y-I mechanism produces a Walrasian allocation (Ekici [8]).

⁷Knuth [10] also proved this equivalence result in an independent study.

Choose a priority-order uniformly at random. Then execute the associated *You request my house - I get your turn mechanism* (in short, Y-I) as follows:⁸ Assign the first agent her most preferred house, the second agent her most preferred house among remaining ones, and so on, until an agent requests the occupied house of an existing tenant. If at that point that existing tenant has already been assigned a house, do not disturb the procedure. Otherwise, update the remainder of the order by inserting that existing tenant to the top and proceed. If at any point a loop forms, it is formed exclusively by a subset of existing tenants who request the occupied houses of one another. In such cases remove the existing tenants in the loop by assigning them the houses they request and proceed.

In a house allocation problem, random Y-I reduces to random-priority and CFRD reduces to CFRE. Therefore, the equivalence result in Abdulkadiroğlu and Sönmez [1] is a corollary of our more general equivalence result.

Our equivalence result contributes to our understanding of the links between popular mechanisms in divisible and indivisible resource allocation problems. In allocating indivisible resources, priority-order based mechanisms—random Y-I and random-priority, are popular, perhaps owing to their simplicity in formulation. Procedurally, however, they cannot be given interpretations from a market point of view. On the other hand, CFRE and CFRD have explicit market interpretations by formulation: First, they induce private ownership exchange markets via probabilistic equal-division, and then they produce Walrasian allocations in the resulting exchange markets. The way they proceed is analogous to the Walrasian Mechanism from equal-division, the popular mechanism in allocating divisible resources. Thus the equivalence results in our paper and in [1] expose that, although this is not explicit by formulation, the popular divisible and indivisible resource allocation mechanisms in the literature are indeed analogous.

The rest of the paper is organized as follows. The next subsection briefly mentions the related literature. Section 2 introduces the model, describes random Y-I and CFRD, and presents our equivalence result. The proof is bijective and fairly involved, which we cover

⁸See Sönmez and Ünver [20] for a characterization of this mechanism.

exclusively in Section 3 (where we present an alternative specification of IATTC and explore its properties). Section 4 concludes the paper. We present the proof of Theorem 1 in the Appendix.

Related Literature

In addition to [1], there are several other papers with equivalence results in the literature:

Sönmez and Ünver [19] showed that the TTC based mechanism in (1) in Section 1 is equivalent to the following priority-order based mechanism: Choose a priority-order by ordering k newcomers at the top uniformly at random and placing $n - k$ existing tenants at the bottom (in any order); then execute the Y-I mechanism defined by that priority-order.

In two recent papers random-priority has been shown to be equivalent to certain mechanisms that execute TTC based upon “inheritance tables.” An inheritance table is a collection of orderings of agents. Each ordering relates to a house. While TTC is executed, an agent points to her most preferred house in the market (as usual) and a house points to the agent in the market who is ordered highest in its ordering. Pathak and Sethuraman [12] show that, if TTC is executed based upon a randomly generated inheritance table where every agent is included in every ordering, the resulting mechanism is equivalent to random-priority. They also extend this equivalence to the houses-with-quotas case (e.g., a public school with a quota of q can be assigned to q students). Also, they show that, in the houses-with-quotas case, if TTC is executed based upon a randomly generated inheritance table where the ordering for a house with quota q includes only q agents and each agent is included in the ordering of only one house, the resulting mechanism is again equivalent to random-priority. Carroll [6] later showed a general equivalence result that implies and extends the preceding ones.

There is a conceptual difference between IATTC based CFRD and the above-mentioned TTC based mechanisms. The execution of TTC in these mechanisms is based upon a randomly generated inheritance table, while the execution of IATTC is based upon randomly generated “inheritor relationships between agents.” This innovation in CFRD promises a new line of research. Future research papers may study how to execute IATTC in the houses-with-quotas case, or when an existing tenant may initially own multiple houses, which may potentially

lead to the design of other IATTC based mechanisms that are equivalent to variants of random Y-I. The tools that we introduce in Section 3 may become useful in these efforts.

2 House Allocation with Existing Tenants

2.1 Preliminaries

A *house allocation problem with existing tenants* is a five-tuple $\langle A_N, H_V, A_E, H_O, P \rangle$ where

- $A_N : \{a_1, a_2, \dots, a_k\}$ is a finite set of “newcomers”;
- $H_V : \{h_1, h_2, \dots, h_k\}$ is a finite set of “vacant houses”;
- $A_E : \{e_{k+1}, e_{k+2}, \dots, e_n\}$ is a finite set of “existing tenants”;
- $H_O : \{o_{k+1}, o_{k+2}, \dots, o_n\}$ is a finite set of “occupied houses” such that existing tenant e_s owns (or, equivalently, occupies) o_s for $s = k + 1, \dots, n$;
- $P : (P_a)_{a \in A_N \cup A_E}$ is the profile of agents’ *strict* preference relations over the set of houses.⁹

A house allocation problem with existing tenants is a *housing market* if every agent is an existing tenant (i.e., $k = 0$), and it is a *house allocation problem* if every agent is a newcomer (i.e., $k = n$).

We fix A_N, H_V, A_E, H_O throughout the paper, and we denote $A_N \cup A_E$ and $H_V \cup H_O$ respectively by A and H .

We assume that every agent prefers being assigned *any* house to not being assigned a house. For $a \in A$ and $h, h' \in H$ we write $h P_a h'$ if a prefers h to h' . We denote the domain of admissible preference relations by \mathcal{P} (so, $P \in \mathcal{P}^n$). We use R_a to represent the “at least as good as” relation for $a \in A$ derived from P_a (i.e., $h R_a h'$ means $h P_a h'$ or $h = h'$).

An *allocation* $\mu : A \rightarrow H$ is a bijection from the set of agents to the set of houses. We denote the set of admissible allocations by \mathcal{M} .

Given $P \in \mathcal{P}^n$, an allocation $\mu \in \mathcal{M}$ is:

⁹i.e., no agent is indifferent between two distinct houses.

- *Pareto-efficient* if there exists no $\mu' \in \mathcal{M}$ such that $\mu'(a) R_a \mu(a)$ for every $a \in A$ and $\mu'(a) P_a \mu(a)$ for an agent $a \in A$.
- *individually-rational* if $\mu(e_s) R_{e_s} o_s$ for $s = k + 1, \dots, n$.
- *group-rational* if there exists no triplet $\langle C, H^C, \theta \rangle$ where $C \subseteq A_E$; $H^C \subseteq H$ is the set of houses owned by agents in C ; and $\theta : C \rightarrow H^C$ is a bijection such that $\theta(a) R_a \mu(a)$ for every $a \in C$ and $\theta(a) P_a \mu(a)$ for an agent $a \in C$. If there exists such a triplet then C is called a “blocking coalition” and we say that μ is “blocked” by C .

In the context of a housing market a group-rational allocation is also called a *core allocation*.

An allocation $\mu \in \mathcal{M}$ is a *Walrasian allocation* (with transfers) if there exists a non-negative price function $p : H \rightarrow \mathbb{R}^+ \cup \{0\}$ and a non-negative transfer function $tr : A \rightarrow \mathbb{R}^+ \cup \{0\}$ such that

1. the budget function $w : A \rightarrow \mathbb{R}^+ \cup \{0\}$ is given by $w(a) = tr(a)$ for $a \in A_N$ and $w(e_s) = tr(e_s) + p(o_s)$ for $s = k + 1, \dots, n$;
2. $p(\mu(a)) \leq w(a)$ for every $a \in A$;
3. $\sum_{a \in A} tr(a) \leq \sum_{h \in H_V} p(h)$;
4. if $h P_a \mu(a)$ for any $a \in A$ and $h \in H$, then $w(a) < p(h)$.

In words, at a Walrasian allocation, vacant houses are sold in the market and the raised revenue is distributed to agents, existing tenants raise additional revenue by selling their occupied houses, and then agents buy in the market their most preferred affordable houses. Ekici [8] shows that the inequalities in (2) and (3) are binding.

A *random assignment* $\lambda : \mathcal{M} \rightarrow \mathbb{R}$ is a probability distribution over allocations. We denote the domain of admissible random assignments by Λ . Note that for every $\lambda \in \Lambda$,

$$\lambda(\mu) \geq 0 \text{ for every } \mu \in \mathcal{M}, \text{ and } \sum_{\mu \in \mathcal{M}} \lambda(\mu) = 1.$$

A *mechanism* (or, *an allocation rule*) is a systematic way to choose an allocation at any given preference profile. Formally, a “deterministic” mechanism $\varphi^D : \mathcal{P}^n \rightarrow \mathcal{M}$ is a function that maps the domain of admissible preference profiles to the codomain of allocations (so at $P \in \mathcal{P}^n$ its allocation choice is certain), and a “lottery” mechanism $\varphi^L : \mathcal{P}^n \rightarrow \Lambda$ is a function that maps the domain of admissible preference profiles to the codomain of random assignments (so, at $P \in \mathcal{P}^n$ it chooses an allocation randomly based upon $\varphi^L(P)$). We have given these definitions in the context of a house allocation problem with existing tenants, but we will also talk about mechanisms in more restricted domains, such as a housing market or a house allocation problem. Therefore, in what follows, a “mechanism” should be understood as a systematic way to choose an allocation in the context of a well-specified class of problems.

A lottery mechanism is ex-post (Pareto) efficient if it maps every preference profile to a random assignment at which positive probability weights are given to only Pareto-efficient allocations. For a lottery mechanism, the properties of ex-post individually-rationality and ex-post group-rationality are defined accordingly.

In the remainder of the paper we will consider a representative problem $\Pi : \langle A_N, H_V, A_E, H_O, P \rangle$, which stands for the class of house allocation problems with existing tenants.

2.2 Random You request my house - I get your turn

In this subsection we present a lottery mechanism in the context of a house allocation problem with existing tenants. It is derived from the class of “You request my house - I get your turn” (Y-I) mechanisms, introduced by Abdulkadiroğlu and Sönmez [2].

A *priority-order* is a bijection from the set of agents A to the set of numbers $\{1, 2, \dots, n\}$. We denote a generic priority-order by f , and the domain of admissible priority-orders by \mathcal{F} . For instance, if $f(a) = 4$ for $a \in A$ and $f \in \mathcal{F}$, it means that agent a is ordered fourth in the priority-order f .

Each priority-order $f \in \mathcal{F}$ defines a separate Y-I mechanism, which allocates houses to agents at a given preference profile as described below.

The Y-I mechanism defined by $f \in \mathcal{F}$: Assign the agent ordered first in f to her most preferred house; assign the agent ordered second in f to her most preferred house among remaining ones; and so on, until an agent requests the occupied house of an existing tenant. If at that point the existing tenant whose occupied house is requested has already been assigned a house, do not disturb the procedure. Otherwise, update the remainder of the priority-order by inserting that existing tenant to the top and proceed. If at any point a loop forms, it is formed exclusively by a subset of existing tenants who request the occupied houses of one another. In such cases remove all agents in the loop by assigning them the houses they request and proceed.

There are appealing properties of the class of Y-I mechanisms. They are strategy-proof (truthful preference revelation is a dominant strategy) (Abdulkadiroğlu and Sönmez [2]); for any given preference profile, they lead to Pareto-efficient and individually-rational allocations [2], and the set of allocation induced by them coincides with the set of Walrasian allocations (Ekici [8]). The following example demonstrates the workings of a Y-I mechanism.

EXAMPLE 1 Consider a house allocation problem with existing tenants in which the preference profile of agents is as in the following table:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_3	h_1	h_3	h_6	h_4	o_{11}	o_8	o_9	h_2	h_1	o_{10}	h_5
\vdots	\vdots	o_{10}	\vdots	\vdots	o_8	h_1	\vdots	\vdots	o_{11}	\vdots	\vdots
		o_{12}			\vdots	o_{10}			\vdots		
		\vdots				h_7					
						\vdots					

Let the priority order f order agents as $a_1, a_2, e_8, e_{11}, a_4, a_5, e_{10}, e_9, a_3, a_6, e_{12}, a_7$ (so $f(a_1) = 1, \dots, f(a_7) = 12$). The following figure illustrates how the Y-I mechanism specified by f pro-

ceeds.

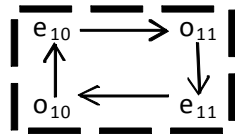
Step 1: No cycles

Step 2: $a_1 \longrightarrow h_3$ (assigned)

THEN

$a_2 \longrightarrow h_1$ (assigned)

Step 3:



each agent in the cycle is assigned the house she points to

Step 4: $e_8 \longrightarrow o_9$ owned by e_9 ; e_9 becomes the agent with highest priority among remaining ones

THEN

$e_9 \longrightarrow h_2$ (assigned)

THEN

$e_8 \longrightarrow o_9$ (assigned)

THEN

$a_4 \longrightarrow h_6$ (assigned)

THEN

$a_5 \longrightarrow h_4$ (assigned)

THEN

$a_3 \longrightarrow o_{12}$ owned by e_{12} ; e_{12} becomes the agent with highest priority among remaining ones

THEN

$e_{12} \longrightarrow h_5$ (assigned)

THEN

$a_3 \longrightarrow o_{12}$ (assigned)

THEN

$a_6 \longrightarrow o_8$ (assigned)

THEN

$a_7 \longrightarrow h_7$ (assigned)

The allocation induced is as follows:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_3	h_1	o_{12}	h_6	h_4	o_8	h_7	o_9	h_2	o_{11}	o_{10}	h_5

◇

In our representative problem Π , let $\mu^{Y-I,f}$ denote the allocation chosen by the Y-I mechanism defined by $f \in \mathcal{F}$, and let $\mathcal{F}^\mu \subseteq \mathcal{F}$ be the subset of priority-orders for which the resulting Y-I mechanisms choose $\mu \in \mathcal{M}$ (i.e., $\mathcal{F}^\mu = \{f \in \mathcal{F} | \mu^{Y-I,f} = \mu\}$).

Despite its other appealing properties, a Y-I mechanism suffers on grounds of fairness. If in a priority-order $f \in \mathcal{F}$ we have $f(a) < f(a')$ for two agents $a, a' \in A$, then the Y-I mechanism defined by f clearly favors a over a' . A natural way to introduce fairness is randomization. That is, one may first choose a priority-order uniformly at random and then execute the Y-I mechanism defined by the chosen priority-order. This is what we call the *random You request my house - I get your turn* mechanism (in short, random Y-I).

Random Y-I: *Choose a priority-order of agents $f \in \mathcal{F}$ uniformly at random, and then allocate houses to agents by executing the Y-I mechanism defined by f .*

Having been derived from the class of Y-I mechanisms, random Y-I is strategy-proof, ex-post efficient, and ex-post group-rational.

In our representative problem Π , let λ^{rY-I} denote the random assignment induced by random Y-I. Then,

$$\lambda^{rY-I}(\mu) = \frac{|\mathcal{F}^\mu|}{n!}.$$

2.3 Core from Random Distribution

In this subsection we introduce an alternative lottery mechanism in the context of a house allocation problem with existing tenants. For this purpose we first recall Gale's reputable *top trading cycles* mechanism (in short, TTC), an allocation rule defined in the context of a housing market that proceeds as follows.

TTC: *Step 1: Let each agent "point" to her most preferred house, and each house "point" to its owner. There exists at least one "cycle," characterized by a list a^1, a^2, \dots, a^j of agents where a^1 points to a^2 's house which points to a^2 ; a^2 points to a^3 's house which*

points to $a^3; \dots; a^{j-1}$ points to a^j 's house which points to a^j ; and a^j points to a^1 's house which points to a^1 . Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones and let each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to and then remove these agents and houses from the market.

TTC is strategy-proof (Roth [13]), and in a housing market it chooses the unique core allocation, which is also the unique Walrasian allocation (Roth and Postlewaite [14]).

We also need to introduce what we call an “inheritors augmented housing market,” which is generated from our representative problem Π .

DEFINITION 1 *From $\Pi : \langle A_N, H_V, A_E, H_O, P \rangle$ an **inheritors augmented housing market** $\Pi^v : \langle A_N, H_V, A_E, H_O, P, v \rangle$ is generated by specifying a bijection $v : A_N \cup A_E \rightarrow H_V \cup I$ such that:*

- $I : \{i_{k+1}, \dots, i_n\}$ is the set of “inheritance rights,” where i_s is the inheritance right associated with existing tenant e_s ;
- agent $a \in A_N \cup A_E$ owns $v(a)$ (which is a vacant house or an inheritance right).

In words, an inheritors augmented housing market is generated from a house allocation problem with existing tenants by distributing to agents vacant houses and inheritance rights associated with existing tenants. Note that in an inheritors augmented housing market it is possible that an existing tenant owns two houses (a vacant house besides her occupied house) and a newcomer owns none (she then owns only an inheritance right).

For $a \in A$ if $v(a) = i_s$, then we call a the “inheritor” of e_s and e_s the “bequeather” of a . If we talk of “bequeathers” of a , the agents we mean by it are a 's bequeather, a 's bequeather's bequeather, and so on. We denote by \mathcal{V} the domain of admissible bijections from $A_N \cup A_E$ to $H_V \cup I$. Note that $|\mathcal{V}| = n!$, and by separately augmenting $n!$ bijections to our representative problem Π we can generate $n!$ distinct inheritors augmented housing markets. If we talk of an

“allocation” in an inheritors augmented housing market, we mean by it, as usual, a bijection from A to H .

The essential component of our alternative lottery mechanism is the “inheritors augmented top trading cycles” (in short, IATTC), a mechanism that operates on an inheritors augmented housing market; as described below, IATTC resembles TTC.

IATTC: *Step 1: Let each agent point to her most preferred house, and each house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to, and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.*

Step $t > 1$: Let each remaining agent point to her most preferred house among remaining ones, and each remaining house point to its owner. There exists at least one cycle. Assign the agents in cycles the houses they point to, and then remove these agents and houses from the market. If an existing tenant in a cycle owns two houses, one of her two houses remains. Then let her remaining house be owned by her inheritor. If her inheritor is an existing tenant who has already been assigned a house, let her remaining house be owned by the inheritor of her inheritor, and so on.

It is simple to verify that IATTC is well-defined. The key observation is that while IATTC operates on an inheritors augmented housing market, any newcomer who does not own a house eventually inherits a house from one of her bequeathers, and so she joins a cycle in which she trades that house. Let a^1 be a newcomer who does not own a house and let a^2 be the existing tenant who is her bequeather. There are two possibilities: either a^2 owns two houses or a^2 has a bequeather—say, a^3 . If the former is true, a^2 joins a cycle in which she trades one of her two houses; when she departs, a^1 inherits the house that remains. If the latter is true, there are again two possibilities: either a^3 owns two houses or she has a bequeather—say, a^4 . If the former is true, a^3 joins a cycle in which she trades one of her

two houses; when she departs, a^1 inherits the house that remains if a^2 has departed earlier; if a^2 has not departed yet, a^2 inherits the remaining house ending up owning two houses (her occupied house and the house she inherited), and after she departs by joining a cycle one of these two houses will again be inherited by a^1 . By iterating similarly, we conclude that a^1 always inherits a house and hence joins a cycle.

The following example demonstrates the workings of IATTC.

EXAMPLE 2 *Consider an inheritors augmented housing market generated from the house allocation problem with existing tenants in Example 1 by distributing to agents vacant houses and inheritance rights as in the following table:*

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7

Thus, the distribution of houses and inheritance rights to agents in this inheritors augmented housing market is as in the following table:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3, o_8	i_9, o_9	i_8, o_{10}	h_6, o_{11}	h_7, o_{12}

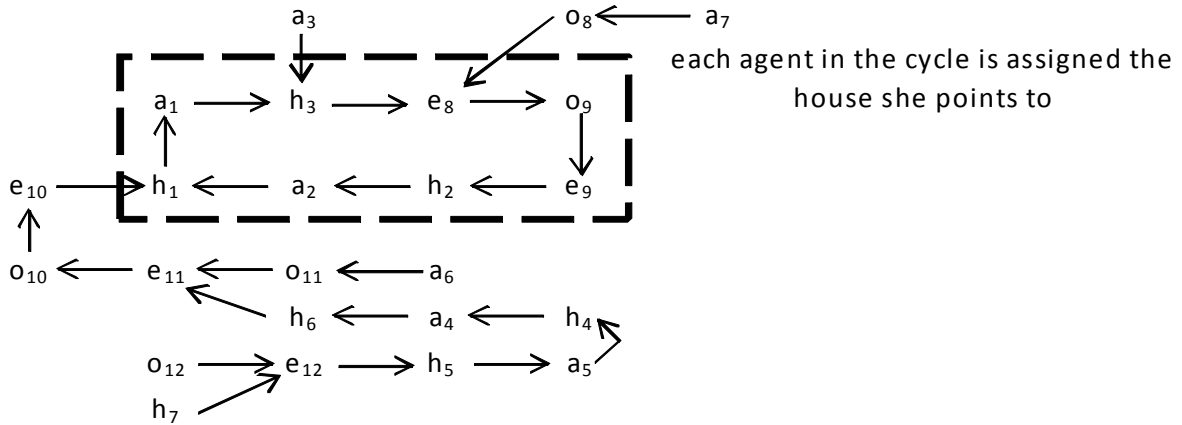
We will illustrate how IATTC proceeds in a series of figures. For visual ease we indicate cycles in the figures in dashed rectangles.

When each house points to its owner and each agent points to her most preferred house,

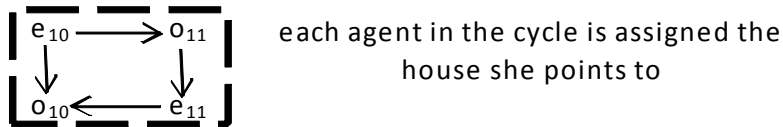
the resulting figure is as follows:

Step 1: No cycles

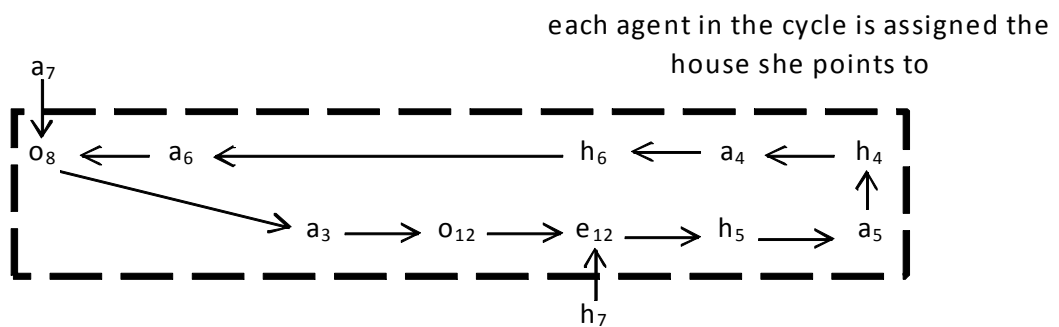
Step 2:



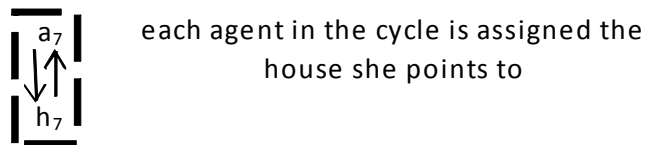
Step 3:



Step 4:



THEN



As prescribed by the cycle in the figure, a_1, e_8, e_9, a_2 are assigned h_3, o_9, h_2, h_1 , respectively. The house o_8 of e_8 remains, which is received by her inheritor e_{10} .

When each remaining house points to its owner and each remaining agent points to her

most preferred house among remaining ones, the resulting figure is as follows:

As prescribed by the cycle in the figure, e_{10} and e_{11} are assigned o_{11} and o_{10} , respectively. The houses h_6 of e_{11} and o_8 of e_{10} remain, which are received by their inheritors a_6 and a_3 , respectively.

When each remaining house points to its owner and each remaining agent points to her most preferred house among remaining ones, the resulting figure is as follows:

As prescribed by the cycle in the figure, $e_{12}, a_5, a_4, a_6, a_3$ are assigned $h_5, h_4, h_6, o_8, o_{12}$, respectively. The house h_7 of e_{12} remains, which is received by her inheritor a_7 .

The last cycle is formed by a_7 and h_7 ; thus, a_7 is assigned h_7 :

The allocation chosen by IATTC is the same as the one we obtained in Example 1. \diamond

In our representative problem Π , let $\mu^{iattc,v}$ denote the allocation chosen by IATTC in Π^v , and let $\mathcal{V}^\mu \subseteq \mathcal{V}$ be the set of bijections from $A_N \cup A_E$ to $H_V \cup I$ such that, in the inheritors augmented housing markets generated by augmenting them to Π , the allocation chosen by IATTC is μ (i.e., $\mathcal{V}^\mu = \{v \in \mathcal{V} | \mu^{iattc,v} = \mu\}$).

We should point out that, in essence, an inheritors augmented housing market generalizes a housing market, and IATTC generalizes TTC. If Π is a housing market (i.e., $k = 0$) and Π^v is an inheritors augmented housing market generated from Π , in Π^v every agent owns exactly one house. Then no house ever remains from an agent who departs from the market, rendering the distribution v without significance, and hence IATTC proceeds just the same as TTC. As it turns out, in this generalization the desirable theoretical properties of TTC are retained by IATTC: It can be shown that IATTC is strategy-proof; also, the allocation it chooses in an inheritors augmented housing market is the unique core allocation if the

“core allocation” notion is properly redefined taking account of the rights of inheritors to the houses of their bequeathers.

DEFINITION 2 *An allocation $\mu : A \rightarrow H$ is a **core allocation** in $\Pi^v : \langle A_N, H_V, A_E, H_O, P, v \rangle$ ($v \in \mathcal{V}$) if there exists no four-tuple $\langle C, H^C, \theta, Claim \rangle$ where $C \subseteq A$, $H^C \subseteq H$, and $\theta : C \rightarrow H^C$ and $Claim : C \rightarrow H^C$ are bijections such that:*

(i) $\theta(a) R_a \mu(a)$ for every $a \in C$ and $\theta(a) P_a \mu(a)$ for an agent $a \in C$;

(ii) for any $a \in C$, a owns $Claim(a)$; or a is the inheritor of an agent $a' \in C$ who owns $Claim(a)$; or a is the inheritor of an agent $a' \in C$ who is the inheritor of another agent $a'' \in C$ who owns $Claim(a)$; or so on.

If there exists such a four-tuple we say that μ is “blocked” by $\langle C, H^C, \theta, Claim \rangle$ and we call C a “blocking coalition.”

The subtle part in the above definition is the *Claim* function. It states that in a blocking coalition every agent needs to claim (bring into the coalition) a distinct house, which should be a house that she or one of her bequeathers owns. In case an agent, say a , claims a house owned by one of her bequeathers, say a'' , inheritors of a'' that are more closely related to her than a should also be in the blocking coalition. That is, if a'' is the bequeather of a' and a' is the bequeather of a , then a' should also be in the blocking coalition. This requirement ensures that the allocation is not blocked to benefit the distant inheritor a at the expense of the more immediate inheritor a' . If every agent owns precisely one house, Definition 2 reduces to the familiar core allocation notion in a housing market.

THEOREM 1 *For $v \in \mathcal{V}$ the allocation $\mu^{iattc,v}$ is the unique core allocation in the inheritors augmented housing market Π^v .*

Proof. See Appendix. ■

We are now ready to introduce our alternative lottery mechanism in the context of a house allocation problem with existing tenants.

Core from Random Distribution: *Distribute n “items”— k vacant houses and $n - k$ inheritance rights associated with $n - k$ existing tenants, to n agents uniformly at random*

(each agent receives exactly one vacant house or one inheritance right). In the generated inheritors augmented housing market, reallocate houses to agents by executing IATTC.

We shortly call this mechanism CFRD. From the theoretical properties of its main component, IATTC, it is not difficult to show that CFRD is strategy-proof, ex-post efficient, and ex-post group-rational.

In our representative problem Π , let λ^{cfrd} denote the random assignment induced by CFRD. Then,

$$\lambda^{cfrd}(\mu) = \frac{|\mathcal{V}^\mu|}{n!}.$$

Theorem 2 presents the main result of our paper. Its proof is bijective and fairly involved, which we cover exclusively in Section 3.

THEOREM 2 *Random Y-I and CFRD are equivalent. That is, for any $\mu \in \mathcal{M}$,*

$$\lambda^{rY-I}(\mu) = \lambda^{cfrd}(\mu).$$

3 The Proof of Theorem 2

In this section we provide an alternative specification of IATTC. As we proceed, we introduce some tools, make certain observations about this alternative specification, and present four lemmas, which help us prove Theorem 2. The proof involves the construction of a bijection as in Abdulkadiroğlu and Sönmez [1], but our construction is fairly more involved due to the presence of existing tenants.

Recall that, in CFRD, first n items (k vacant houses and $n - k$ inheritance rights) are distributed to n agents uniformly at random, and then, in the generated inheritors augmented housing market, houses are reallocated to agents by executing IATTC. In the execution of IATTC an existing tenant is assigned a house by joining a cycle, in which the house that she trades comes from one of two resources. It is either her occupied house, or a house that she receives due to the item that she received in the random distribution (i.e., a vacant house that she received in the random distribution, or a house that is accrued to her because of an

inheritance right that she received in the random distribution). The distinguishing feature of our alternative specification of IATTC is that, it “monitors” the potential benefits to an existing tenant from these two resources by representing her in the exchange market with two copies of her, one of them owning her occupied house, and the other owning the item that she received in the random distribution. This separation allows us to construct a priority-order of agents from the distribution of items to agents. Our construction turns out to be a bijective mapping and leads to the proof of Theorem 2.

From a given inheritors augmented housing market $\Pi^v : \langle A_N, H_V, A_E, H_O, P, v \rangle$, we construct its “ab-representation” $\Pi^{v,ab} : \langle A_N, H_V, A_E^a, A_E^b, H_O, P, v \rangle$ in the following manner:

- We preserve the set of newcomers A_N : $a \in A_N$ owns $v(a)$.
- We replace the set of existing tenants A_E by two disjoint sets, A_E^a and A_E^b : Each existing tenant $e_s \in A_E$ in Π^v is now “represented” in $\Pi^{v,ab}$ by two distinct agents, $a_s \in A_E^a$ who owns $v(e_s)$, and $b_s \in A_E^b$ who owns o_s . The preferences of a_s and b_s are the same as the preferences of e_s . Although technically a_s and b_s are two separate agents, they are both to serve the interests of e_s , and hence we call them the “sisters” of one another.

In the ab-representation, we refer to the agents in $A_N \cup A_E^a$ as “a-type” agents, and to the agents in A_E^b as “b-type” agents. As an illustration, we present below how inheritance rights and houses are distributed to agents in the inheritors augmented housing market in

Example 2 and in its ab-representation:

inheritors augmented housing market in Example 2											
A_N							A_E				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	o_8, h_3	o_9, i_9	o_{10}, i_8	o_{11}, h_6	o_{12}, h_7

↓

ab-representation																
<i>a-type agents</i>							<i>b-type agents</i>									
A_N							A_E					B_E				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7	o_8	o_9	o_{10}	o_{11}	o_{12}

We are now ready to introduce the “ab-representation specification” of IATTC, or, shortly, IATTC^{ab} .

IATTC^{ab}: *Given an inheritors augmented housing market Π^v , construct its ab-representation $\Pi^{v,ab}$. In $\Pi^{v,ab}$, reallocate houses and inheritance rights to a-type and b-type agents by the following iterative procedure:*

Step 0,1: (b-step) Let every remaining house and inheritance right point to its owner. Among remaining agents, let only b-type agents point. A b-type agent $b_s \in A_E^b$ points to her most preferred house among remaining ones if $a_s \in A_E^a$ has not been assigned a house yet, and she points to i_s if a_s has already been assigned a house. If there exists one or more cycles, remove the agents in cycles by assigning them the houses and inheritance rights they point to.

Step 0,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Step 1,0: (a-step) Let every remaining house and inheritance right point to its owner. Now, let every remaining agent (both a-type and b-type) point. A newcomer $a \in A_N$ points to her most preferred house among remaining ones. Of two sister agents $a_s \in A_E^a$ and $b_s \in A_E^b$, if neither has been assigned a house yet, let them both point to their most

preferred house among remaining ones; if one of them has been assigned a house before, let the remaining one point to i_s . There exists at least one cycle. Remove the agents in cycles by assigning them the houses and inheritance rights they point to.

Step 1,1: (b-step) Same as Step 0,1.

Step 1,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Step t,0: (a-step) Same as Step 1,0.

Step t,1: (b-step) Same as Step 0,1.

Step t,r: (b-step) Same as Step 0,1. (Continue until there exists no cycles)

Stop when the procedure assigns every a-type and b-type agent a house or an inheritance right. Then, in Π^v , let the houses assigned to agents be as follows: For a newcomer $a \in A_N$, the house assigned to her is the house the above procedure assigns $a \in A_N$ in $\Pi^{v,ab}$; and for an existing tenant $e_s \in A_E$, the house assigned to her is the house the above procedure assigns $a_s \in A_E^a$ or $b_s \in A_E^b$ (the procedure assigns a house to only one of them, the other is assigned i_s).

Notice that IATTC^{ab} proceeds just like TTC , by identifying cycles and then carrying out the trades in cycles, but it gives precedence to the trades in cycles that involve only b-type agents. At prior b-steps, trades are carried out in cycles that involve only b-type agents, and when no such cycle remains, IATTC^{ab} moves to an a-step at which it carries out the trades in cycles that involve both a-type and b-type agents.

Two observations are useful to better understand the design of IATTC^{ab} .

† **Observation 1:** Suppose for an existing tenant $e_s \in A_E$ in Π^v ($v \in \mathcal{V}$) it happens that $v(e_s) \in H_V$ (so, e_s owns two houses, $v(e_s)$ and o_s). Notice how IATTC and IATTC^{ab} proceed analogously:

When IATTC is executed in Π^v , at initial steps $v(e_s)$ and o_s point to e_s and e_s points to her most preferred house among remaining ones; when IATTC^{ab} is executed in $\Pi^{v,ab}$, at initial steps $v(e_s)$ and o_s respectively point to a_s and b_s (the agents that represent e_s), and a_s and b_s point to e_s 's most preferred house among remaining ones.

In IATTC's execution in Π^v , when e_s joins a cycle in which she exchanges $v(e_s)$ or o_s , in parallel to that, in IATTC^{ab}'s execution in $\Pi^{v,ab}$, a_s or b_s joins the analogous cycle in which she exchanges $v(e_s)$ or o_s .

In IATTC's execution in Π^v , the house that remains from e_s is given to her inheritor; in IATTC^{ab}'s execution in $\Pi^{v,ab}$, the analogous thing happens: The remaining house ($v(e_s)$ or o_s) points to the remaining sister agent (a_s or b_s); the remaining sister agent points to i_s ; i_s points to the a-type or b-type agent that represents the inheritor of e_s ; and hence, in essence, the remaining house is transferred to the inheritor of e_s .

For $v \in \mathcal{V}$ let $\mu^{ab,v}$ denote the allocation chosen by IATTC^{ab} in Π^v . Given Observation 1 the following lemma is evident.

LEMMA 1 *IATTC and IATTC^{ab} are equivalent. That is, for any $v \in \mathcal{V}$,*

$$\mu^{iattc,v} = \mu^{ab,v}.$$

† **Observation 2:** *In IATTC^{ab}, at a b-step only b-type agents point (to houses or inheritance rights), and thus:*

- (i) *a cycle at a b-step consists of only b-type agents and occupied houses;*
- (ii) *a-type agents, vacant houses, and inheritance rights are part of the cycles at a-steps, but a cycle at an a-step may also include b-type agents and occupied houses.*

The separation of the steps in IATTC^{ab} as a-steps and b-steps is fundamental to our proof of Theorem 2. In the following example we demonstrate the workings of IATTC^{ab}.

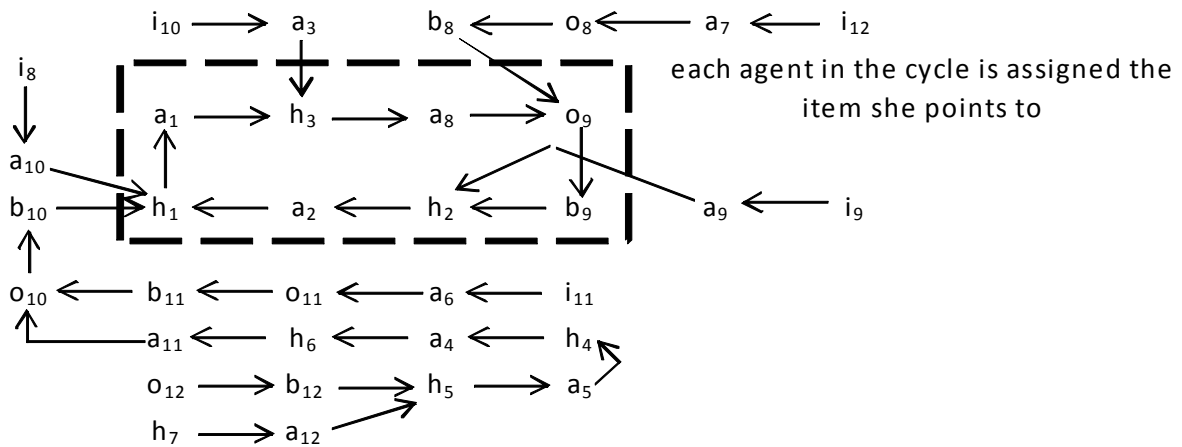
EXAMPLE 3 *Consider the ab-representation of the inheritors augmented housing market in Example 2. The table below presents the distribution of houses and inheritance rights to a-type and b-type agents:*

a-type agents												b-type agents				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_1	h_2	i_{10}	h_4	h_5	i_{11}	i_{12}	h_3	i_9	i_8	h_6	h_7	o_8	o_9	o_{10}	o_{11}	o_{12}

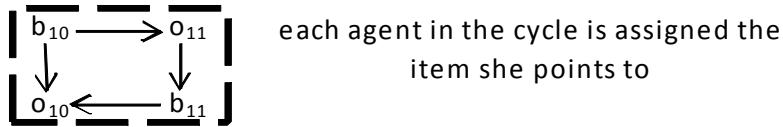
We illustrate in a series of figures below how IATTC^{ab} proceeds. While looking into the figures, recall that remaining houses and inheritance rights point (to their owners) at both a-steps and b-steps; remaining b-type agents also point (to houses and inheritance rights) at both a-steps and b-steps; but remaining a-type agents point (to houses and inheritance rights) only at a-steps. For visual ease we indicate cycles in the figures in dashed rectangles.

Step 1: No cycles

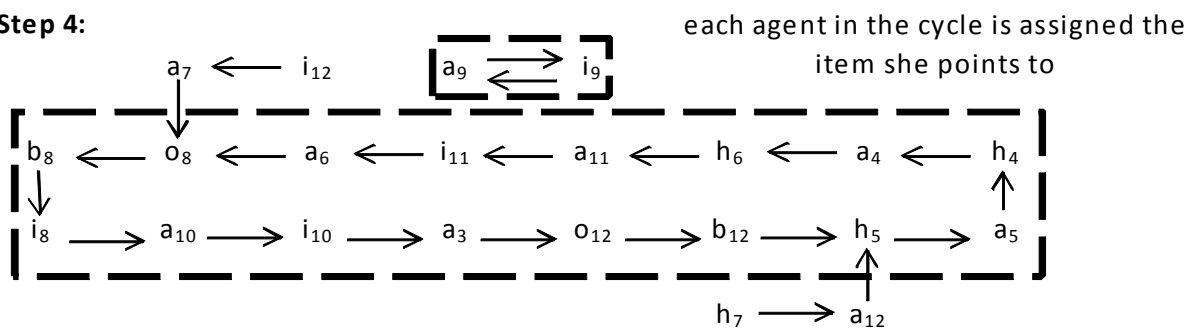
Step 2:



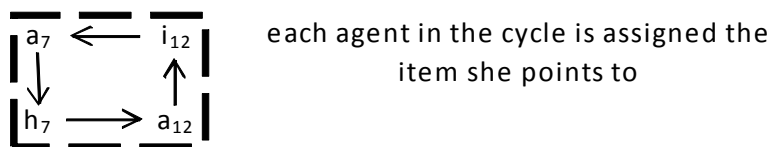
Step 3:



Step 4:



THEN



There are no cycles at Step 0,1. IATTC^{ab} proceeds to Step 1,0.

There is one cycle at Step 1,0. As prescribed by the cycle, a_1, a_8, b_9, a_2 are assigned h_3, o_9, h_2, h_1 , respectively. IATTC^{ab} proceeds to Step 1,1.

There is one cycle at Step 1,1. As prescribed by the cycle, b_{10} and b_{11} are assigned o_{11} and o_{10} , respectively. IATTC^{ab} proceeds to Step 1,2.

At Step 1,2 there are two remaining b-type agents, b_8 and b_{12} , who respectively point to i_8 and h_5 . The resulting figure would be the same as the preceding figure except that the cycle in the figure is removed. There are no cycles and thus IATTC^{ab} proceeds to Step 2,0.

There are two cycles at Step 2,0. As prescribed by the cycles, $a_3, b_{12}, a_5, a_4, a_{11}, a_6, b_8, a_{10}, a_9$ are assigned $o_{12}, h_5, h_4, h_6, i_{11}, o_8, i_8, i_{10}, i_9$, respectively. IATTC^{ab} proceeds to Step 2,1.

Since there is no remaining b-type agent, there are no cycles at Step 2,1, and the mechanism proceeds to Step 3,0.

There is one cycle at Step 3,0. As prescribed by the cycle, a_7 and a_{12} are assigned h_6 and i_{12} , respectively, and the procedure terminates.

The houses the procedure assigns to a-type and b-type agents, and the implied assignments to agents made by IATTC^{ab} in the inheritors augmented housing market, are as follows:

<i>Assignments of a-type agents</i>												<i>Assignments of b-type agents</i>				
a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	b_8	b_9	b_{10}	b_{11}	b_{12}
h_3	h_1	o_{12}	h_6	h_4	o_8	h_7	o_9	i_9	i_{10}	i_{11}	i_{12}	i_8	h_2	o_{11}	o_{10}	h_5

↓

<i>Assignments of newcomers and existing tenants</i>											
a_1	a_2	a_3	a_4	a_5	a_6	a_7	e_8	e_9	e_{10}	e_{11}	e_{12}
h_3	h_1	o_{12}	h_6	h_4	o_8	h_7	o_9	h_2	o_{11}	o_{10}	h_5

◇

In what follows we introduce some tools about IATTC^{ab} , and based upon these tools we make certain observations.

‡ **TOOL 1, sets defined by the order of cycle groups:** In the execution of IATTC^{ab} in $\Pi^{v,ab}$ ($v \in \mathcal{V}$), houses and inheritance rights are assigned to a-type and b-type agents in a well-defined order of cycle groups. Based upon this order of cycle groups, we define below certain sets of agents, houses, and inheritance rights:

- $A_{t,r}^v$: the set of a-type and b-type agents that are assigned a house or an inheritance right in a cycle at Step t,r .
- $A_0^v = A_{0,1}^v \cup A_{0,2}^v \cup \dots$ and $A_t^v = A_{t,0}^v \cup A_{t,1}^v \cup \dots$ for $t \geq 1$.
- $H_{t,r}^v$: the set of houses assigned to agents in $A_{t,r}^v$.
- $H_0^v = H_{0,1}^v \cup H_{0,2}^v \cup \dots$ and $H_t^v = H_{t,0}^v \cup H_{t,1}^v \cup \dots$ for $t \geq 1$.
- $I_{t,0}^v$: the set of inheritance rights assigned to agents in $A_{t,0}^v$ for $t \geq 1$. (Recall from Observation 2 (i) that in the cycles at b-steps there are no inheritance rights.)

‡ **TOOL 2, a-blocks at an a-step:** In the execution of IATTC^{ab} in $\Pi^{v,ab}$ ($v \in \mathcal{V}$), we define an “a-block” at an a-step Step $t,0$ ($t \geq 1$) as an ordered list $bl_t^v(a) : (a, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y)$ (or $bl_t^v(a) : (a, y)$) where

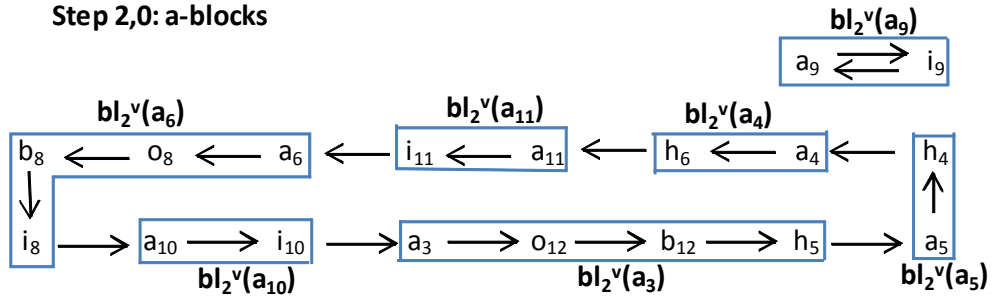
- $a \in A_{t,0}^v \cap (A_N \cup A_E^a)$; $y \in (H_{t,0}^v \cap H_V) \cup I_{t,0}^v$;

– $k + 1 \leq \pi_p \leq n$ for $p = 1, \dots, q$;

– at Step $t,0$ a points to o_{π_1} , b_{π_1} points to o_{π_2} , \dots , and b_{π_q} points to y (if $bl_t^v(a) : (a, y)$ then simply a points to y).

We call a the “source” and y the “sink” of the a-block $bl_t^v(a)$. With some abuse of notation, we denote the set $\{a, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y\}$ (or $\{a, y\}$) also by $bl_t^v(a)$.

More simply, an a-block is a segment of a cycle that arises at an a-step in the execution of IATTC^{ab}. It starts with the only a-type agent of that a-block, and ends with a vacant house or an inheritance right owned by another a-type agent. At an a-step sinks of a-blocks (vacant houses and inheritance rights) point to the sources of a-blocks (a-type agents), and hence the cycles form. As an illustration, in the figure below we indicate in enclosed boxes the a-blocks at Step 2,0 in Example 3.



The following observation summarizes our preceding discussion on a-blocks.

† **Observation 3:** *In the execution of IATTC^{ab} in $\Pi^{v,ab}$ ($v \in V$), the cycles that arise at an a-step Step $t,0$ ($t \geq 1$) consist of a-blocks. The sinks of a-blocks point to the sources of a-blocks, and hence the cycles form. So,*

$$(i) \quad \bigcup_{a \in A_{t,0}^v \cap (A_N \cup A_E^a)} bl_t^v(a) = A_{t,0}^v \cup H_{t,0}^v \cup I_{t,0}^v;$$

$$(ii) \quad \text{and for } a, a' \in A_{t,0}^v \cap (A_N \cup A_E^a) \text{ and } a \neq a', bl_t^v(a) \cap bl_t^v(a') = \emptyset.$$

The following is another simple observation pertaining to a-blocks, which later proves useful.

† **Observation 4:** *Suppose we are given the list of sets $(A_{j,0}^v \cap (A_N \cup A_E^a))_{j=1}^t$ but we do not know $v \in V$. (That is, we are given the sets of a-type agents that are assigned houses*

at Step 1,0, Step 2,0, \dots , Step $t,0$ when $IATTC^{ab}$ is executed in $\Pi^{v,ab}$.) Then, we can determine

(i) to which house or inheritance right a remaining agent points to up to Step $t+1,0$;

(ii) the assignments made by $IATTC^{ab}$ up to Step $t+1,0$;

and we can identify

(iii) the a -block $bl_j^v(a)$ for any $a \in A_{j,0}^v \cap (A_N \cup A_E^a)$ and $j \in \{1, \dots, t\}$.

Explanation: The execution of $IATTC^{ab}$ in $\Pi^{v,ab}$ at Step 0,1, Step 0,2, and so on, is independent of v . Then, to which house or inheritance right a remaining agent points to, and the assignments made, can be determined up to Step 1,0. For the subsequent steps, we can iteratively apply the following arguments for $j=1, \dots, t$:

Given the assignments made prior to Step $j,0$, we know to which house or inheritance right a remaining agent points to at Step $j,0$. So, for any $a \in A_{j,0}^v \cap (A_N \cup A_E^a)$, we can identify $bl_j^v(a)$. But then we can also determine the assignments made at Step $j,0$: Each agent in an a -block is assigned the house she points to in that a -block.

Given the assignments made at Step $j,0$ and prior to it, the execution of $IATTC^{ab}$ in $\Pi^{v,ab}$ at Step $j,1$, Step $j,2$, and so on, is independent of v . Then, to which house or inheritance right a remaining agent points to, and the assignments made, can be determined for Step $j,1$, Step $j,2$, and so on.

‡ **TOOL 3, chains at an a-step:** Consider the execution of $IATTC^{ab}$ in $\Pi^{v,ab}$ ($v \in V$). The elements of $A_{t+1,0}^v \cup H_{t+1,0}^v \cup I_{t+1,0}^v$ ($t \geq 1$), which form the cycle(s) at Step $t+1,0$, form at Step $t,0$ what we call “chains.” Formally, a chain at Step $t,0$ is an ordered list $ch_t^v(x_1) : (x_1, y_1, \dots, x_q, y_q)$ ($t \geq 1, q \geq 1$) where

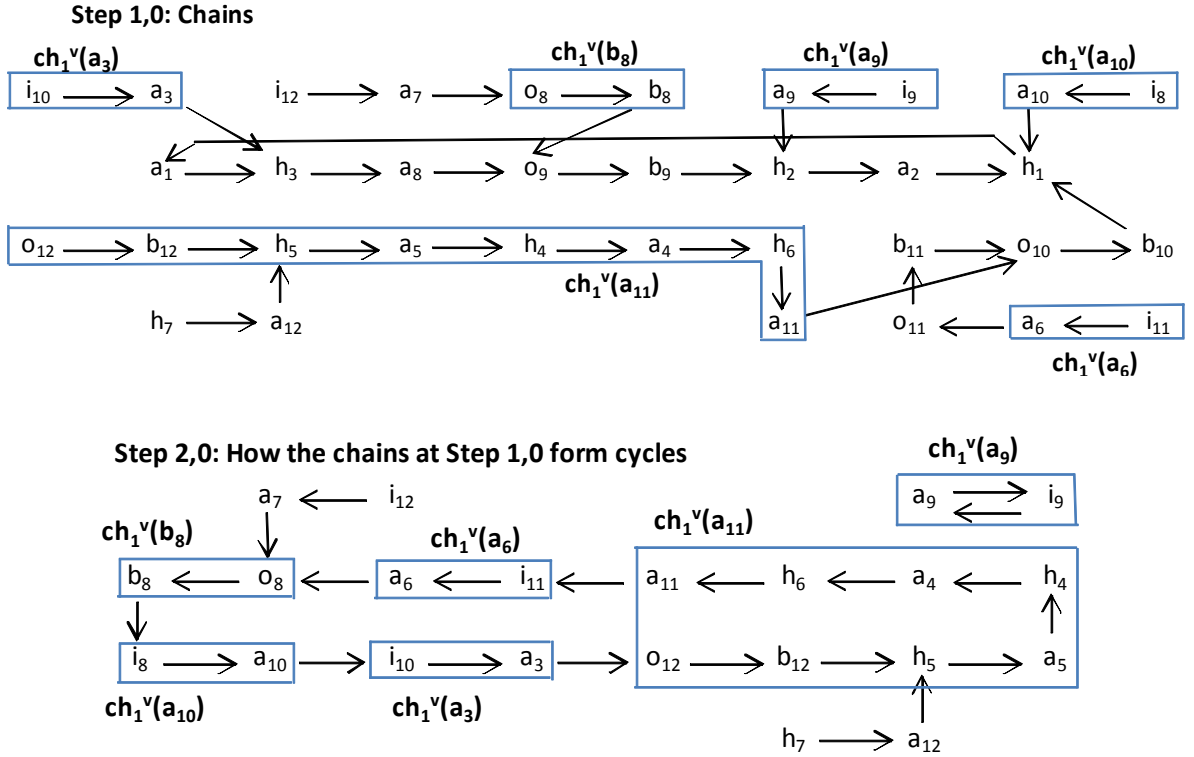
– $x_p \in A_{t+1,0}^v$ and $y_p \in H_{t+1,0}^v \cup I_{t+1,0}^v$ for $p = 1, \dots, q$;

– at Step $t,0$ x_1 points to a house or an inheritance right in $H_t^v \cup I_{t,0}^v$; y_1 points to x_1 (i.e., x_1 owns y_1); x_2 points to y_1 ; \dots ; y_q points to x_q (i.e., x_q owns y_q);

– there exists no $x_{q+1} \in A_{t+1,0}^v$ who points to y_q at Step $t,0$.

We call x_1 the “head” of $ch_t^v(x_1)$ and y_q the “tail” of $ch_t^v(x_1)$. With some abuse of notation, we denote the set $\{x_1, y_1, \dots, x_q, y_q\}$ also by $ch_t^v(x_1)$.

More simply, a chain at Step $t,0$ is a connected (by pointers) elements of $A_{t+1,0}^v \cup H_{t+1,0}^v \cup I_{t+1,0}^v$. At Step $t+1,0$, the heads of chains at Step $t,0$ point to the tails of chains at Step $t,0$, hence the cycles form. As an illustration, in the figures below we indicate in enclosed boxes the chains at Step 1,0 in Example 3, and how they form the cycles at Step 2,0.



Notice that at Step $t,0$ ($t \geq 1$) the head of a chain, which by definition points to a house or an inheritance right, indeed always points to a house: By construction of $IATTC^{ab}$, an inheritance right i_s is pointed by only one agent— a_s or b_s , whoever is assigned later. But if the head of a chain at Step $t,0$ points to an inheritance right, it means she is not assigned that inheritance right, which would be a contradiction. (As an illustration, notice that in the second preceding figure all heads of chains point to houses.)

The following observation summarizes our preceding discussion on chains.

† **Observation 5:** *In the execution of $IATTC^{ab}$ in $\Pi^{v,ab}$ ($v \in V$), let X be the set of heads of chains at Step $t,0$ ($t \geq 1$). Then,*

(i) a head of a chain $x \in X$ points to a house in $H_{t,0}^v$ at Step $t,0$, and to a house or an inheritance right in $H_{t+1,0}^v \cup I_{t+1,0}^v$ at Step $t+1,0$;

(ii) an agent $a \in A_{t+1,0}^v / X$ points to the same house or inheritance right in $H_{t+1,0}^v \cup I_{t+1,0}^v$ at Step $t,0$ and Step $t+1,0$.

For the chains at Step $t,0$, at Step $t+1,0$ their heads point to their tails, and hence the cycles at Step $t+1,0$ form. Then,

$$(iii) \bigcup_{x \in X} ch_t^v(x) = A_{t+1,0}^v \cup H_{t+1,0}^v \cup I_{t+1,0}^v;$$

$$(iv) \text{ and for } x, x' \in X \text{ and } x \neq x', ch_t^v(x) \cap ch_t^v(x') = \emptyset.$$

From the head to the tail of a chain, we call the a-type agent ordered first the “a-head” of the chain, and the a-type agent ordered last the “a-tail” of the chain. For instance, looking into the second preceding figure above, the a-head and a-tail of $ch_1^v(a_{11})$ are respectively a_{11} and a_5 . Looking into that figure, also note that in a chain (i) the head and a-head can be the same (e.g., $ch_1^v(a_{11})$); (ii) there may be only one a-type agent and so its a-head and a-tail can be the same (e.g., $ch_1^v(a_3)$); (iii) there may be no a-type agents, in which case we call it an “empty chain” (e.g., $ch_1^v(b_8)$).

‡ TOOL 4, the chain-order o_{ch}^v of a-type agents

For $v \in \mathcal{V}$, the “chain-order” $o_{ch}^v : A_N \cup A_E^a \rightarrow \{1, 2, \dots, n\}$ of a-type agents is a bijection that orders a-type agents according to the following three rules:

Chain-order Rule 1: In the chain-order o_{ch}^v , order a-type agents in $A_{1,0}^v \cap (A_N \cup A_E^a)$ before a-type agents in $A_{2,0}^v \cap (A_N \cup A_E^a)$; order a-type agents in $A_{2,0}^v \cap (A_N \cup A_E^a)$ before a-type agents in $A_{3,0}^v \cap (A_N \cup A_E^a)$; and so on.

Chain-order Rule 2: In the chain-order o_{ch}^v , order a-type agents in $A_{1,0}^v \cap (A_N \cup A_E^a)$ in order of the indices of vacant houses and inheritance rights that they are assigned at v .

Chain-order Rule 3: In the chain-order o_{ch}^v , order the a-type agents in $A_{t+1,0}^v \cap (A_N \cup A_E^a)$ ($t \geq 1$) in the following manner: Consider the chains at Step $t,0$. Order the a-type agents in a non-empty chain from its a-head to its a-tail, consecutively. Order non-empty

chains in order of the indices of vacant houses and inheritance rights that their a-tails are assigned at v .

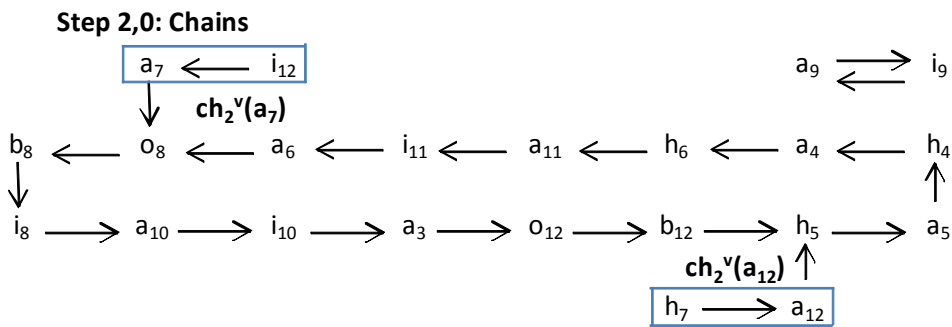
As an illustration, consider the execution of $IATTC^{ab}$ in the ab-representation of the inheritors augmented housing market in Example 3:

By Chain-order Rule 1, in o_{ch}^v the a-type agents assigned at Step 1,0 (i.e., a_1, a_2, a_8) are ordered before the a-type agents assigned at Step 2,0 (i.e., $a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6$), who are ordered before the a-type agents assigned at Step 3,0 (i.e., a_{12}, a_7).

By Chain-order Rule 2, in o_{ch}^v the three a-type agents assigned at Step 1,0 are ordered as a_1, a_2, a_8 . (Note that a_1, a_2, a_8 own respectively h_1, h_2, h_3 , whose indices are respectively 1, 2, 3.)

By Chain-order Rule 3 and looking into the second preceding figure above, the order of non-empty chains at Step 1,0 is $ch_1^v(a_{11}), ch_1^v(a_{10}), ch_1^v(a_9), ch_1^v(a_3), ch_1^v(a_6)$ (a-tails of these chains own respectively $h_5, i_8, i_9, i_{10}, i_{11}$; indices are respectively 5, 8, 9, 10, 11), and hence the chain-order of a-type agents in $A_{2,0}^v$ is $a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6$.

The figure below shows the chains at Step 2,0 in Example 3, formed by the elements of $A_{3,0}^v \cup H_{3,0}^v \cup I_{3,0}^v$.



By Chain-order Rule 3 and looking into the preceding figure, the order of non-empty chains at Step 2,0 is $ch_2^v(a_{12}), ch_2^v(a_7)$ (a-tails of these chains respectively own h_7, i_{12} ; indices are respectively 7 and 12), and hence the chain-order of a-type agents in $A_{3,0}^v$ is a_{12}, a_7 .

Therefore, the chain-order of a-type agents that we obtain in Example 3 is:

$$o_{ch}^v : \quad \underbrace{a_1, a_2, a_8}_{A_{1,0}^v \cap (A_N \cup A_E^a)} \quad \underbrace{a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6}_{A_{2,0}^v \cap (A_N \cup A_E^a)} \quad \underbrace{a_{12}, a_7}_{A_{3,0}^v \cap (A_N \cup A_E^a)}$$

‡ **TOOL 5, the chain priority-order f_{ch}^v of newcomers and existing tenants:** For $v \in \mathcal{V}$, from the chain-order o_{ch}^v of a-type agents, we derive the “chain priority-order” $f_{ch}^v \in \mathcal{F}$ of newcomers and existing tenants in a straightforward way. Simply set $f_{ch}^v(a) = o_{ch}^v(a)$ for $a \in A_N$, and $f_{ch}^v(e_s) = o_{ch}^v(a_s)$ for $e_s \in A_E$.

For instance, the chain order given above, and the chain priority-order derived from it, are as follows:

$$\begin{aligned} o_{ch}^v & : a_1, a_2, a_8, a_{11}, a_4, a_5, a_{10}, a_9, a_3, a_6, a_{12}, a_7 \\ f_{ch}^v & : a_1, a_2, e_8, e_{11}, a_4, a_5, e_{10}, e_9, a_3, a_6, e_{12}, a_7 \end{aligned}$$

Observe that f_{ch}^v is precisely the same priority-order as the one considered in Example 1. Also, recall that the allocations chosen in Example 3 by IATTC^{ab} , and in Example 1 by the Y-I mechanism defined by f_{ch}^v , are the same. Lemma 2 states that this holds in general.

LEMMA 2 For any $v \in \mathcal{V}$,

$$\mu^{ab,v} = \mu^{Y-I, f_{ch}^v}.$$

Proof. It is plain to see that the lemma holds once the following observation is made. Considering the execution of the Y-I mechanism defined by f_{ch}^v in Π , and the execution of IATTC^{ab} in $\Pi^{v,ab}$, a loop in the former one corresponds to a cycle at a b-step in the latter one, and an out-of-loop assignment made in the former one corresponds to an a-block at an a-step in the latter one. We elaborate below.

In the execution of the Y-I mechanism defined by f_{ch}^v in Π , let it be the turn of agent $a \in A$ to request a house. Also, in the execution of IATTC^{ab} in $\Pi^{v,ab}$, let Step $t,0$ be when the a-type agent that represents a is assigned a house or an inheritance right. When a requests a house, one of the following five cases occurs:

(1) Agent a requests a house that triggers the formation of one or more loops (if a is an existing tenant, she may also be part of one of these loops). In the execution of IATTC^{ab} in

$\Pi^{v,ab}$, these loops correspond to certain cycles that arise at b -steps prior to Step $t,0$: The agents in the cycles are the b -type agents that represent the existing tenants in the loop, and they are assigned the same houses at $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v} .

(2) Agent a requests a vacant house $h \in H_V$. In the execution of IATTC^{ab} in $\Pi^{v,ab}$, this case corresponds to the a -block $bl_t^v(a) : (a, h)$ at Step $t,0$. Agent a is assigned h at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v} .

(3) Agent a requests the occupied house o_s of $e_s \in A_E$, who has already been assigned a house before. In the execution of IATTC^{ab} , this case corresponds to the a -block $bl_t^v(a) : (a, o_s, b_s, i_s)$ at Step $t,0$. Agent a is assigned o_s at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v} .

(4) Agent a requests the occupied house o_{π_1} of $e_{\pi_1} \in A_E$; e_{π_1} moves to the top of the remainder of the priority-order and requests the occupied house o_{π_2} of $e_{\pi_2} \in A_E$; \dots ; e_{π_q} moves to the top of the remainder of the priority-order and requests a vacant house $h \in H_V$. In the execution of IATTC^{ab} in $\Pi^{v,ab}$, this case corresponds to the a -block $bl_t^v(a) : (a, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, h)$ at Step $t,0$. Agents $a, e_{\pi_1}, \dots, e_{\pi_q}$ are assigned the houses $o_{\pi_1}, \dots, o_{\pi_q}, h$, respectively, at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v} .

(5) The same thing happens as in (4) except that e_{π_q} requests the occupied house o_s of $e_s \in A_E$, who has already been assigned a house before. In the execution of IATTC^{ab} in $\Pi^{v,ab}$, this case corresponds to the a -block $bl_t^v(a) : (a, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, o_s, b_s, i_s)$ at Step $t,0$. Agents $a, e_{\pi_1}, \dots, e_{\pi_q}$ are assigned the occupied houses $o_{\pi_1}, \dots, o_{\pi_q}, o_s$, respectively, at both $\mu^{ab,v}$ and μ^{Y-I,f_{ch}^v} . ■

The following lemma states that if the executions of IATTC^{ab} in two inheritors augmented housing markets induce the same chain priority-order, then a -type agents join cycles at the same a -steps for the two inheritors augmented housing markets.

LEMMA 3 For $v_1, v_2 \in \mathcal{V}$ if $f_{ch}^{v_1} = f_{ch}^{v_2}$, then $A_{t,0}^{v_1} \cap (A_N \cup A_E^a) = A_{t,0}^{v_2} \cap (A_N \cup A_E^a)$ for every $t \geq 1$.

Proof. If $f_{ch}^{v_1} = f_{ch}^{v_2}$, then $\mu^{ab,v_1} = \mu^{ab,v_2}$ (by Lemma 1 and Lemma 2). Also, $o_{ch}^{v_1} = o_{ch}^{v_2}$ (by definition). Let $\delta : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be the bijection such that $o_{ch}^{v_1}(a_{\delta(s)}) = o_{ch}^{v_2}(a_{\delta(s)}) = s$ for $s = 1, 2, \dots, n$. The proof is by induction.

Base Case:

Suppose $A_{1,0}^{v_1} \cap (A_N \cup A_E^a) \neq A_{1,0}^{v_2} \cap (A_N \cup A_E^a)$. W.l.o.g. let

$$A_{1,0}^{v_1} \cap (A_N \cup A_E^a) = \{a_{\delta(1)}, a_{\delta(2)}, \dots, a_{\delta(\alpha)}\},$$

$$A_{1,0}^{v_2} \cap (A_N \cup A_E^a) = \{a_{\delta(1)}, a_{\delta(2)}, \dots, a_{\delta(\alpha)}, a_{\delta(\alpha+1)}, \dots, a_{\delta(\beta)}\} \text{ (i.e., } \beta > \alpha \geq 1).$$

We present our arguments in four steps:

(1) Show that $bl_1^{v_1}(a_{\delta(s)}) = bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, 2, \dots, \alpha$:

The execution of IATTC^{ab} prior to Step 1,0 is independent of v_1 and v_2 (i.e., the same assignments are made prior to Step 1,0). Then, for Π^{v_1} and Π^{v_2} the same a-type agents, b-type agents, houses, and inheritance rights remain at Step 1,0, and remaining a-type and b-type agents point to the same houses and inheritance rights. So, $bl_1^{v_1}(a_{\delta(s)}) = bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, 2, \dots, \alpha$.

Let $bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y)$ where $k+1 \leq \pi_p \leq n$ for $p = 1, \dots, q$ and $y \in H_V \cup I$. (The arguments are essentially the same if $bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, y)$).

(2) Show that $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$:

Since $y \in bl_1^{v_2}(a_{\delta(\alpha+1)})$, we get $y \notin bl_1^{v_2}(a_{\delta(s)})$ for $s = 1, \dots, \alpha$ (by Observation 3 (ii)), and hence $y \notin bl_1^{v_1}(a_{\delta(s)})$ for $s = 1, \dots, \alpha$. Then, $y \notin H_{1,0}^{v_1} \cup I_{1,0}^{v_1}$ (see Observation 3 (i)). Since $y \in H_V \cup I$, we get $y \notin H_{1,r}^{v_1}$ also for $r \geq 1$ (by Observation 2 (i)). Then, $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$.

(3) Show that $bl_2^{v_1}(a_{\delta(\alpha+1)}) = bl_1^{v_2}(a_{\delta(\alpha+1)})$:

From $bl_1^{v_2}(a_{\delta(\alpha+1)})$ we know that for Π^{v_2} at Step 1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1} ; o_{π_1} points to b_{π_1} ; \dots ; and b_{π_q} points to y . Since the execution of IATTC^{ab} prior to Step 1,0 is independent of v_1 and v_2 , also for Π^{v_1} at Step 1,0 $a_{\delta(\alpha+1)}$ points to o_{π_1} ; o_{π_1} points to b_{π_1} ; \dots ; and b_{π_q} points to y . Since $y \notin H_1^{v_1} \cup I_{1,0}^{v_1}$, for Π^{v_1} this sequence remains unaffected until Step 2,0. Then, $bl_2^{v_1}(a_{\delta(\alpha+1)}) = bl_1^{v_2}(a_{\delta(\alpha+1)})$.

(4) Find a contradiction:

Since in $A_{2,0}^{v_1} \cap (A_N \cup A_E^a)$ the a-type agent who comes first in $o_{ch}^{v_1}$ is $a_{\delta(\alpha+1)}$, at Step 1,0 $a_{\delta(\alpha+1)}$ should be the a-head of a non-empty chain. In this chain either $a_{\delta(\alpha+1)}$ is the head, or a b-type agent in $bl_2^{v_1}(a_{\delta(\alpha+1)})$ is the head, say b_{π_j} for some $j \in \{1, 2, \dots, q\}$.

If $a_{\delta(\alpha+1)}$ is the head of the chain, then at μ^{ab, v_1} she is not assigned her most preferred

house in $H \setminus H_0^{v_1}$. But for Π^{v_2} since $a_{\delta(\alpha+1)}$ is assigned at Step 1,0, at μ^{ab,v_2} she is assigned her most preferred house in $H \setminus H_0^{v_2}$ ($= H \setminus H_0^{v_1}$), which contradicts that $\mu^{ab,v_1} = \mu^{ab,v_2}$.

If b_{π_j} is the head of the chain, then from

$$bl_1^{v_2}(a_{\delta(\alpha+1)}) : (a_{\pi(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y),$$

in the execution of $IATTC^{ab}$ in Π^{v_2} , b_{π_j} points at Step 1,0 to $o_{\pi_{j+1}}$ (or y). From the fact that in the execution of $IATTC^{ab}$ in Π^{v_1} at Step 1,0 b_{π_j} is the head of a chain, by Observation 5 (i) she points at Step 1,0 to a house not in $bl_2^{v_1}(a_{\pi(\alpha+1)})$ ($= bl_1^{v_2}(a_{\delta(\alpha+1)})$), which contradicts that for Π^{v_1} and Π^{v_2} at Step 1,0 remaining agents point to same houses an inheritance rights.

Inductive Step: (The arguments are exactly parallel to the base case. For the sake of completeness, we reproduce them below, where changes have been made as necessary).

Suppose $A_{j,0}^{v_1} \cap (A_N \cup A_E^a) = A_{j,0}^{v_2} \cap (A_N \cup A_E^a)$ for $j = 1, \dots, t$ but $A_{t+1,0}^{v_1} \cap (A_N \cup A_E^a) \neq A_{t+1,0}^{v_2} \cap (A_N \cup A_E^a)$. W.l.o.g. let

$$A_{t+1,0}^{v_1} \cap (A_N \cup A_E^a) = \{a_{\delta(l)}, a_{\delta(l+1)}, \dots, a_{\delta(\alpha)}\},$$

$$A_{t+1,0}^{v_2} \cap (A_N \cup A_E^a) = \{a_{\delta(l)}, a_{\delta(l+1)}, \dots, a_{\delta(\alpha)}, a_{\delta(\alpha+1)}, \dots, a_{\delta(\beta)}\} \text{ (i.e., } \beta > \alpha \geq l).$$

We present our arguments in four steps:

(1) Show that $bl_{t+1}^{v_1}(a_{\delta(s)}) = bl_{t+1}^{v_2}(a_{\delta(s)})$ for $s = l, l+1, \dots, \alpha$:

By Observation 4, for Π^{v_1} and Π^{v_2} the same assignments are made by $IATTC^{ab}$ prior to Step $t+1,0$; the same a-type agents, b-type agents, houses, and inheritance rights remain at Step $t+1,0$; remaining a-type and b-type agents point to the same houses and inheritance rights; and $bl_{t+1}^{v_1}(a_{\delta(s)}) = bl_{t+1}^{v_2}(a_{\delta(s)})$ for $s = l, l+1, \dots, \alpha$.

Let $bl_{t+1}^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y)$ where $k+1 \leq \pi_p \leq n$ for $p = 1, \dots, q$ and $y \in H_V \cup I$. (The arguments are essentially the same if $bl_{t+1}^{v_2}(a_{\delta(\alpha+1)}) : (a_{\delta(\alpha+1)}, y)$).

(2) Show that $y \notin H_{t+1}^{v_1} \cup I_{t+1,0}^{v_1}$:

Since $y \in bl_{t+1}^{v_2}(a_{\delta(\alpha+1)})$, we get $y \notin bl_{t+1}^{v_2}(a_{\delta(s)})$ for $s = l, \dots, \alpha$ (by Observation 3 (ii)), and hence $y \notin bl_{t+1}^{v_1}(a_{\delta(s)})$ for $s = l, \dots, \alpha$. Then, $y \notin H_{t+1,0}^{v_1} \cup I_{t+1,0}^{v_1}$ (see Observation 3 (i)). Since $y \in H_V \cup I$, we get $y \notin H_{t+1,r}^{v_1}$ also for $r \geq 1$ (by Observation 2 (i)). Then, $y \notin H_{t+1}^{v_1} \cup I_{t+1,0}^{v_1}$.

(3) Show that $bl_{t+2}^{v_1}(a_{\delta(\alpha+1)}) = bl_{t+1}^{v_2}(a_{\delta(\alpha+1)})$:

From $bl_{t+1}^{v_2}(a_{\delta(\alpha+1)})$ we know that for Π^{v_2} at Step $t+1,0$ $a_{\delta(\alpha+1)}$ points to o_{π_1} ; o_{π_1} points to b_{π_1} ; \dots ; and b_{π_q} points to y . Given that $A_{j,0}^{v_1} \cap (A_N \cup A_E^a) = A_{j,0}^{v_2} \cap (A_N \cup A_E^a)$ for $j = 1, \dots, t$ the execution of IATTC^{ab} prior to Step $t+1,0$ is the same for Π^{v_1} and Π^{v_2} (see Observation 4). Then, also for Π^{v_1} at Step $t+1,0$ $a_{\delta(\alpha+1)}$ points to o_{π_1} ; o_{π_1} points to b_{π_1} ; \dots ; and b_{π_q} points to y . Since $y \notin H_{t+1}^{v_1} \cup I_{t+1,0}^{v_1}$, for Π^{v_1} this sequence remains unaffected until Step $t+2,0$. Then, $bl_{t+2}^{v_1}(a_{\delta(\alpha+1)}) = bl_{t+1}^{v_2}(a_{\delta(\alpha+1)})$.

(4) Find a contradiction:

Since in $A_{t+2,0}^{v_1} \cap (A_N \cup A_E^a)$ the a-type agent who comes first in $o_{ch}^{v_1}$ is $a_{\delta(\alpha+1)}$, at Step $t+1,0$ $a_{\delta(\alpha+1)}$ should be the a-head of a non-empty chain. In this chain either $a_{\delta(\alpha+1)}$ is the head, or a b-type agent in $bl_{t+2}^{v_1}(a_{\delta(\alpha+1)})$ is the head, say b_{π_j} for some $j \in \{1, 2, \dots, q\}$.

If $a_{\delta(\alpha+1)}$ is the head of the chain, then at μ^{ab,v_1} she is not assigned her most preferred house in $H \setminus (H_0^{v_1} \cup H_1^{v_1} \cup \dots \cup H_t^{v_1})$. But for Π^{v_2} since $a_{\delta(\alpha+1)}$ is assigned at Step $t+1,0$, at μ^{ab,v_2} she is assigned her most preferred house in $H \setminus (H_0^{v_2} \cup H_1^{v_2} \cup \dots \cup H_t^{v_2}) (= H \setminus (H_0^{v_1} \cup H_1^{v_1} \cup \dots \cup H_t^{v_1}))$, which contradicts that $\mu^{ab,v_1} = \mu^{ab,v_2}$.

If b_{π_j} is the head of the chain, then from

$$bl_{t+1}^{v_2}(a_{\delta(\alpha+1)}) : (a_{\pi(\alpha+1)}, o_{\pi_1}, b_{\pi_1}, \dots, o_{\pi_q}, b_{\pi_q}, y),$$

in the execution of IATTC^{ab} in Π^{v_2} , b_{π_j} points at Step $t+1,0$ to $o_{\pi_{j+1}}$ (or y). From the fact that in the execution of IATTC^{ab} in Π^{v_1} at Step $t+1,0$ b_{π_j} is the head of a chain, by Observation 5 (i) she points at Step $t+1,0$ to a house not in $bl_{t+2}^{v_1}(a_{\pi(\alpha+1)}) (= bl_{t+1}^{v_2}(a_{\delta(\alpha+1)}))$, which contradicts that for Π^{v_1} and Π^{v_2} at Step $t+1,0$ remaining agents point to same houses an inheritance rights. ■

LEMMA 4 If $f_{ch}^{v_1} = f_{ch}^{v_2}$ for $v_1, v_2 \in \mathcal{V}$, then $v_1 = v_2$.

Proof. Let $f_{ch}^{v_1}$ ($= f_{ch}^{v_2}$) be given but not v_1 . By Lemma 3 $f_{ch}^{v_1}$ uniquely identifies the sets $A_{t,0}^{v_1} \cap (A_N \cup A_E^a) (= A_{t,0}^{v_2} \cap (A_N \cup A_E^a))$ for $t = 1, 2, \dots$ (i.e., to identify them we do not need v_1 and v_2). Also, from $f_{ch}^{v_1}$ we can derive $o_{ch}^{v_1}$ ($= o_{ch}^{v_2}$) (by definition). We will show that, from $A_{t,0}^{v_1} \cap (A_N \cup A_E^a)$ for $t = 1, 2, \dots$ and $o_{ch}^{v_1}$, we can uniquely identify v_1 (and hence also v_2), which proves the lemma.

Let $\delta : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ be the bijection such that $o_{ch}^{v_1}(a_{\delta(s)}) = s$ for $s = 1, 2, \dots, n$. By Observation 4 (ii), from the sets $A_{t,0}^{v_1} \cap (A_N \cup A_E^a)$ for $t = 1, 2, \dots$ we can determine the a-blocks at every a-step. The proof is in two parts.

(I) Let the a-blocks at Step 1,0 be

$$\begin{aligned} bl_1^{v_1}(a_{\delta(1)}) & : (a_{\delta(1)}, \dots, y^1), \\ bl_1^{v_1}(a_{\delta(2)}) & : (a_{\delta(2)}, \dots, y^2), \\ & \vdots \\ bl_1^{v_1}(a_{\delta(\alpha)}) & : (a_{\delta(\alpha)}, \dots, y^\alpha). \end{aligned}$$

By Chain-order Rule 2, at v_1 the a-type agents $a_{\delta(1)}, \dots, a_{\delta(\alpha)}$ are assigned the vacant houses and inheritance rights y^1, \dots, y^α in order of the indices of vacant houses and inheritance rights. Since this is well-defined, we can uniquely identify how y^1, \dots, y^α are assigned to $a_{\delta(1)}, \dots, a_{\delta(\alpha)}$ at v_1 .

(II) For $t \geq 1$ let the a-blocks at Step t+1,0 be

$$\begin{aligned} bl_{t+1}^{v_1}(a_{\delta(m_0)}) & : (a_{\delta(m_0)}, \dots, y^{m_0}), \\ bl_{t+1}^{v_1}(a_{\delta(m_0+1)}) & : (a_{\delta(m_0+1)}, \dots, y^{m_0+1}), \\ & \vdots \\ bl_{t+1}^{v_1}(a_{\delta(m_1-1)}) & : (a_{\delta(m_1-1)}, \dots, y^{m_1-1}) \\ bl_{t+1}^{v_1}(a_{\delta(m_1)}) & : (a_{\delta(m_1)}, \dots, y^{m_1}) \\ & \vdots \\ & \vdots \\ bl_{t+1}^{v_1}(a_{\delta(m_q-1)}) & : (a_{\delta(m_q-1)}, \dots, y^{m_q-1}) \\ bl_{t+1}^{v_1}(a_{\delta(m_q)}) & : (a_{\delta(m_q)}, \dots, y^{m_q}) \\ & \vdots \\ bl_{t+1}^{v_1}(a_{\delta(\beta)}) & : (a_{\delta(\beta)}, \dots, y^\beta) \end{aligned}$$

(i.e., $1 \leq m_0 < m_1 < \dots < m_q \leq \beta$)

such that $a_{\delta(m_0)}, \dots, a_{\delta(m_q)}$ are a-heads of chains at Step $t,0$. (By Observation 5 (i), we can determine the heads of chains at Step $t,0$. They are the agents in $A_{t+1,0}^{v_1}$ who point to a house in $H_{t,0}^v$ at Step $t,0$, and to a house or an inheritance right in $H_{t+1,0}^v \cup I_{t+1,0}^v$ at Step $t+1,0$. Then we can also determine the a-heads of chains at Step $t,0$. An a-type agent $a \in A_{t+1,0}^{v_1} \cap (A_N \cup A_E^a)$ is the a-head of a chain at Step $t,0$ if an agent in $bl_{t+1}^{v_1}(a)$ is the head of a chain at Step $t,0$.)

Then, the a-type agents $a_{\delta(m_0)}, \dots, a_{\delta(m_1-1)}$ are in the same chain at Step $t,0$, and at v_1 , by Chain-order Rule 3, y^{m_0+1} is assigned to $a_{\delta(m_0)}$, y^{m_0+2} is assigned to $a_{\delta(m_0+1)}$, \dots , and y^{m_1-1} is assigned to $a_{\delta(m_1-2)}$. Similarly, $a_{\delta(m_1)}, \dots, a_{\delta(m_2-1)}$ are in the same chain at Step $t,0$, and at v_1 , by Chain-order Rule 3, y^{m_1+1} is assigned to $a_{\delta(m_1)}$, y^{m_1+2} is assigned to $a_{\delta(m_1+1)}$, \dots , and y^{m_2-1} is assigned to $a_{\delta(m_2-2)}$; and so on.

Then, at v_1 the a-tails of chains at Step $t,0$ (i.e., $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \dots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$) are assigned the remaining vacant houses and inheritance rights (i.e., $y^{m_0}, y^{m_1}, \dots, y^{m_q}$). By Chain-order Rule 3 these houses and inheritance rights are assigned to $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \dots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$ in order of their indices. Since this is well-defined, we can also uniquely identify how $y^{m_0}, y^{m_1}, \dots, y^{m_q}$ are assigned to $a_{\delta(m_1-1)}, a_{\delta(m_2-1)}, \dots, a_{\delta(m_q-1)}, a_{\delta(\beta)}$ at v_1 . ■

Given Lemma 2 and Lemma 4 the proof of Theorem 2 is easy.

Proof of Theorem 2. Consider the mapping $f_{ch} : \mathcal{V} \rightarrow \mathcal{F}$ such that $f_{ch}(v) = f_{ch}^v$ for $v \in \mathcal{V}$. By Lemma 4 f_{ch} is an injection. By the facts that f_{ch} is an injection and $|\mathcal{V}| = |\mathcal{F}| = n!$, f_{ch} is a bijection. By Lemma 2 and the fact that f_{ch} is a bijection, we get $\lambda^{Y-I} = \lambda^{cfrd}$. ■

4 Conclusion

We studied the mixed ownership case of the indivisible resource allocation problem, also known as the “house allocation problem with existing tenants.” A mechanism due to Abdulkadiroğlu and Sönmez [2] for this class of problems is random Y-I, which is strategy-proof, ex-post efficient, and ex-post group-rational. We proposed in this context a market-based alternative mechanism, CFRD, which can also be shown to be strategy-proof, ex-post efficient, and ex-post group-rational. There are interesting methodological and theoretical parallels

between CFRD and the Walrasian Mechanism from equal-division, the popular rule used in divisible resource allocation problems. We obtain that CFRD and random Y-I are equivalent mechanisms. In the public ownership case, Abdulkadiroğlu and Sönmez [1] showed the equivalence of “random-priority” and CFRE. Our result generalizes theirs to the mixed ownership setting. The equivalence results in our paper and in theirs expose that, although this is not explicit by formulation, there is a hidden methodological analogy between the popular divisible and indivisible resource allocation mechanisms in the literature. These mechanisms also have interesting theoretical similarities.

In two recent papers, Pathak and Sethuraman [12] and Carroll [6] show the equivalence of random-priority to certain mechanisms that execute TTC based upon randomly generated “inheritance tables.” The key component of CFRD, IATTC, however, executes based upon randomly generated “inheritor relationships between agents.” This innovation promises a new line of research. Future research papers may study how to execute IATTC in problems where an object can be assigned to multiple agents (e.g., in the school choice setting), or when an existing tenant may initially own multiple houses. This line of research may potentially lead to the design of other IATTC based lottery mechanisms that are equivalent to variants of random Y-I. The tools that we introduced in Section 3 may become useful in these efforts.

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APPENDIX

Proof of Theorem 1.

While IATTC operates on Π^v , cycles appear in an order. First, a group of cycles form—call Group 1; agents in these cycles are assigned the houses that they point to; then they depart from the market and houses that remain are inherited by those agents that are still in the market. Then, a second group of cycles forms—call Group 2, and the execution similarly proceeds.

Let A and H be partitioned into $\{A_s\}_{s=1}^T$ and $\{H_s\}_{s=1}^T$ according to cycle groups: A_s and H_s are respectively the sets of agents and houses that join a cycle in Group s for $s = 1, \dots, T$.

Let $\# : A \cup H \rightarrow \{1, 2, \dots, T\}$ be the function such that $\#(x) = s$ if $x \in A_s \cup H_s$. (It specifies to which group of cycles a house or an agent belongs.)

Let $Points : A \rightarrow H$ be the function such that for an agent $a \in A$, $Points(a)$ is the house that she trades (the house that points to her) in the cycle that she joins. Clearly,

$$\#(a) = \#(Points(a)) \quad \forall a \in A.$$

The proof is in two parts:

(I) for any $\mu \in \mathcal{M}$ if $\mu \neq \mu^{iattc,v}$, then μ is not a core allocation in Π^v :

Suppose $\mu \neq \mu^{iattc,v}$ but μ is a core allocation in Π^v .

If $\mu(a) \neq \mu^{iattc,v}(a)$ for an agent $a \in A_1$, then a finds $\mu(a)$ less preferable than $\mu^{iattc,v}(a)$ (because $\mu^{iattc,v}(a)$ is a 's most preferred house in H). Then μ is clearly blocked by the four-tuple $\langle A_1, H_1, \theta, Claim \rangle$ where $\theta(a) = \mu^{iattc,v}(a)$ and $Claim(a) = Points(a)$ for every $a \in A_1$. Then we should have $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1$.

Given that $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1$, if $\mu(a) \neq \mu^{iattc,v}(a)$ for an agent $a \in A_2$, then a finds $\mu(a)$ less preferable than $\mu^{iattc,v}(a)$ (because $\mu^{iattc,v}(a)$ is a 's most preferred house in $H \setminus H_1$ and $\mu(a) \in H \setminus H_1$). Then μ is clearly blocked by the four-tuple $\langle A_1 \cup A_2, H_1 \cup H_2, \theta, Claim \rangle$ where $\theta(a) = \mu^{iattc,v}(a)$ and $Claim(a) = Points(a)$ for every $a \in A_1 \cup A_2$. Then we should have $\mu(a) = \mu^{iattc,v}(a)$ for every $a \in A_1 \cup A_2$.

If we iterate similarly we conclude that $\mu = \mu^{iattc,v}$, which is a contradiction.

(II) $\mu^{iattc,v}$ is a core allocation in Π^v :

Suppose $\mu^{iattc,v}$ is blocked by a four-tuple $\langle C, H^C, \theta, Claim \rangle$.

While IATTC operates on Π^v , whenever an agent is assigned a house, that house is her most preferred house among remaining ones. Therefore, for $a \in A$ if $\theta(a) R_a \mu^{iattc,v}(a)$, then $\#(\theta(a)) \leq \#(\mu^{iattc,v}(a))$, and if $\theta(a) P_a \mu^{iattc,v}(a)$, then $\#(\theta(a)) < \#(\mu^{iattc,v}(a))$. Then,

$$\sum_{h \in H^C} \#(h) < \sum_{a \in C} \#(a). \quad (\star)$$

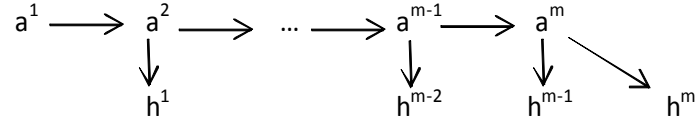
By Definition 2 (ii), agents and houses in $C \cup H^C$ can be partitioned into subsets according to the houses that they claim. Such a subset consists of a list of agents $a^1, a^2, \dots, a^m \subseteq C$ and a list of houses $h^1, h^2, \dots, h^m \subseteq H^C$ where

$$\bigcup_{a \in \{a^1, \dots, a^m\}} Claim(a) = \{h^1, \dots, h^m\},$$

and which adopts one of the following three structures.

STRUCTURE 1: a^1 is the inheritor of a^2 , a^2 is the inheritor of a^3 , \dots , a^{m-1} is the inheritor of a^m ; a^2 owns h^1 , a^3 owns h^2 , \dots , a^{m-1} owns h^{m-2} , a^m owns h^{m-1} and h^m . A graphical representation, in which agents point to their bequeathers and to the houses they

own, is as follows:



While IATTC operates on Π^v , in the cycles that they join, a^m trades a house in $\{h^{m-1}, h^m\}$, a^{m-1} trades a house in $\{h^{m-2}, h^{m-1}, h^m\}, \dots, a^2$ trades a house in $\{h^1, \dots, h^m\}$. Then,

$$\{Points(a^2), Points(a^3), \dots, Points(a^m)\} \subset \{h^1, h^2, \dots, h^m\},$$

and let $\tilde{h} = \{h^1, h^2, \dots, h^m\} \setminus \{Points(a^2), Points(a^3), \dots, Points(a^m)\}$.

Then a^1 trades in the cycle that she joins either \tilde{h} or a house that she owns which is not in H^C (in which case a^1 is an existing tenant who trades in that cycle her occupied house, which is not in H^C). If the former holds, we get

$$\bigcup_{a \in \{a^1, \dots, a^m\}} Points(a) = \{h^1, \dots, h^m\},$$

$$\text{and so } \sum_{h \in \{h^1, \dots, h^m\}} \#(h) = \sum_{a \in \{a^1, \dots, a^m\}} \#(a).$$

If the latter holds, then \tilde{h} joins a cycle after a^1 joins a cycle, and hence $\#(\tilde{h}) > \#(a^1)$.

Then, we get

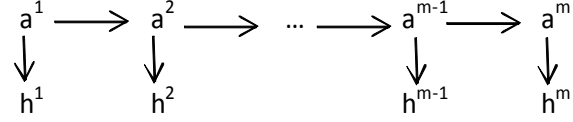
$$\sum_{h \in \{h^1, \dots, h^m\}} \#(h) > \sum_{a \in \{a^1, \dots, a^m\}} \#(a).$$

In either case, for Structure 1, we get

$$\sum_{h \in \{h^1, \dots, h^{m+1}\}} \#(h) \geq \sum_{a \in \{a^1, \dots, a^m\}} \#(a).$$

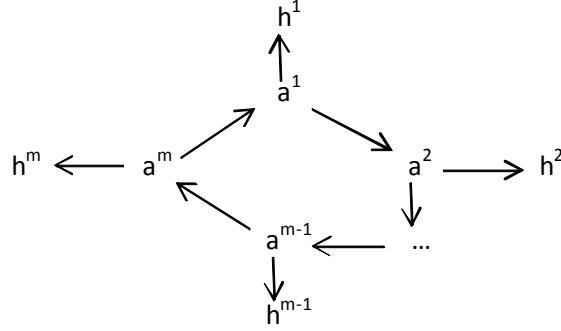
STRUCTURE 2: a^1 is the inheritor of a^2 , a^2 is the inheritor of a^3 , \dots , a^{m-1} is the inheritor of a^m ; a^1 owns h^1 , a^2 owns h^2 , \dots , a^{m-1} owns h^{m-1} , and a^m owns h^m . A graphical representation, in which agents point to their bequeathers and to the houses they own, is as

follows:



While IATTC operates on Π^v , an agent joins a cycle before or at the same time as a house that she owns. Then, $\#(h^s) \geq \#(a^s)$ for $s = 1, 2, \dots, m$. Note that, according to Definition 2, agents under this structure claim the houses that they own.

STRUCTURE 3: a^1 is the inheritor of a^2 , a^2 is the inheritor of a^3 , \dots , a^{m-1} is the inheritor of a^m , a^m is the inheritor of a^1 ; a^1 owns h^1 , a^2 owns h^2 , \dots , a^m owns h^m . A graphical representation, in which agents point to their bequeathers and to the houses they own, is as follows:



By the same argument as in Structure 2, we get $\#(h^s) \geq \#(a^s)$ for $s = 1, 2, \dots, k$. Note that, according to Definition 2, an agent under this structure does not necessarily claim the house that she owns.

From the arguments in Structure 1, Structure 2, and Structure 3, we get,

$$\sum_{h \in H^C} \#(h) \geq \sum_{a \in C} \#(a),$$

which contradicts (★). ■