Engineering Systems for Allocating Public Goods

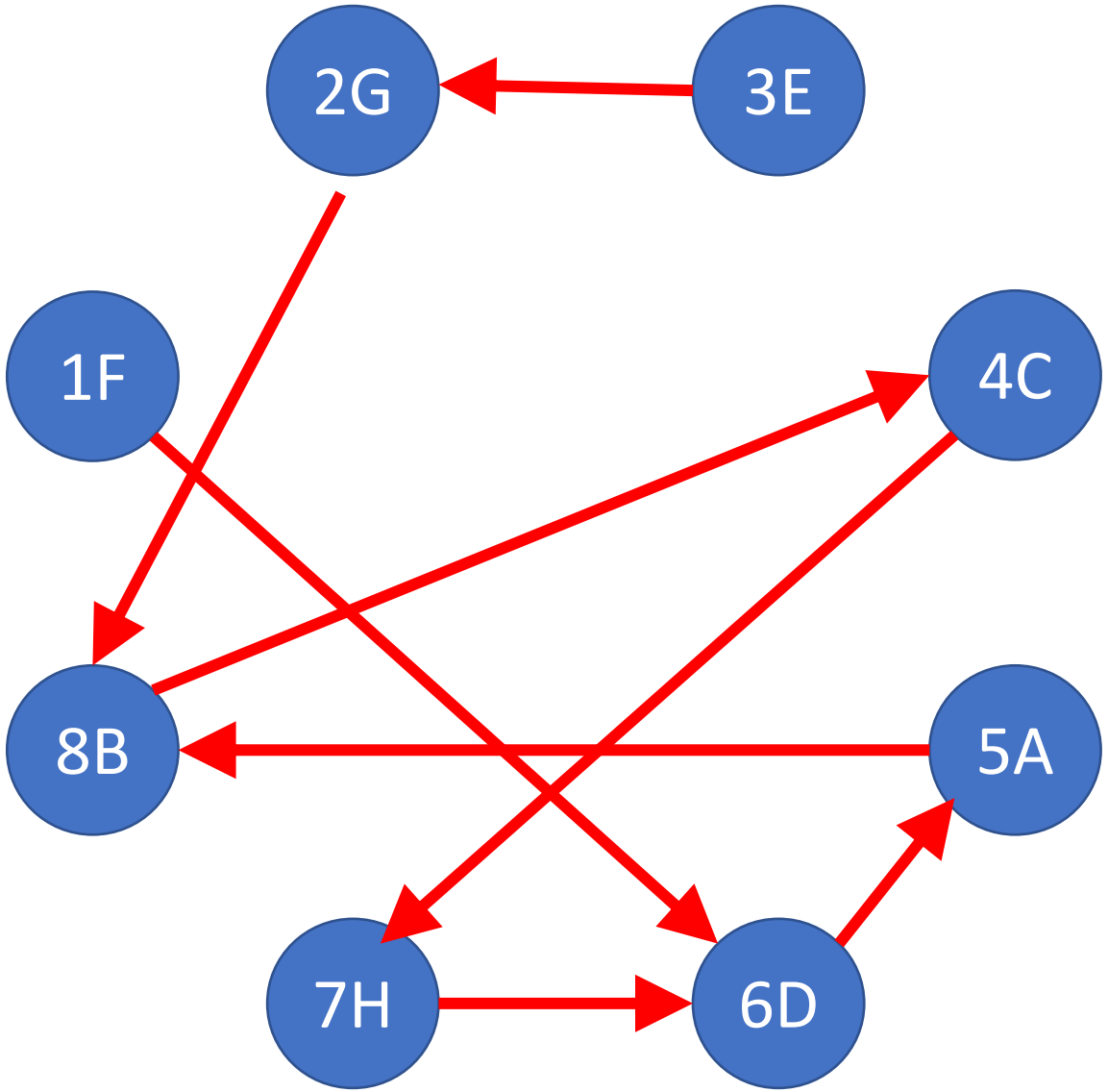
Lecture 3: Fairness

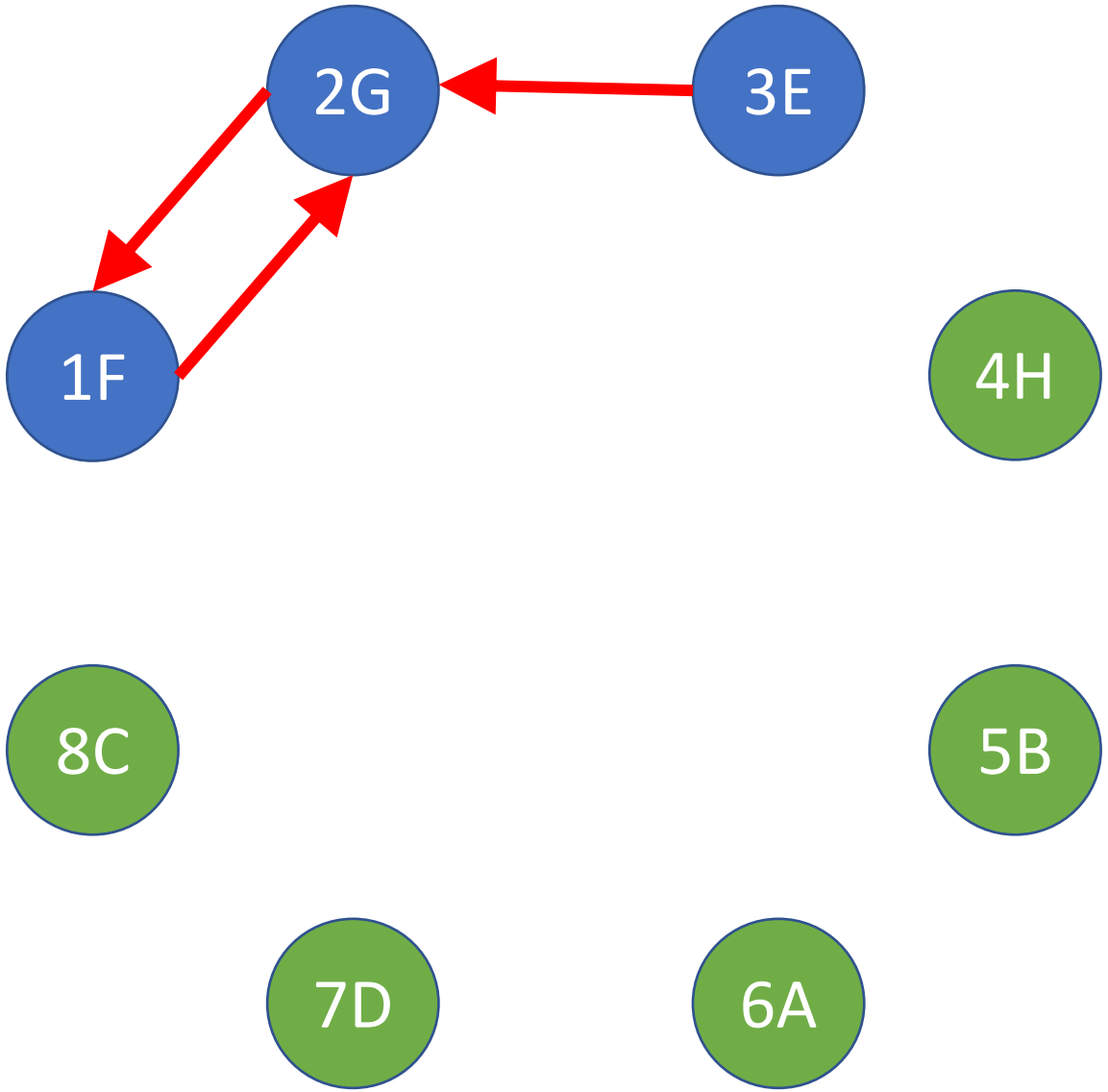
Plan for Today

1. Homework and Concept Review
2. “Fair” Versions of Serial Dictatorship and Top Trading Cycles
3. Fairness definition and other fair mechanisms.

Top Trading Cycles HW Problem

1	2	3	4	5	6	7	8
D	B	G	H	B	A	D	C
G	D	F	D	C	E	F	E
B	C	A	E	G	G	A	D
E	F	C	A	E	H	G	G
F	G	E	G	F	C	B	B
C	E	H	B	A	F	C	F
H	H	D	C	H	B	H	A
A	A	B	F	D	D	E	H


2G
3E
1F
4C
8B
5A
7H
6D


2G
3E
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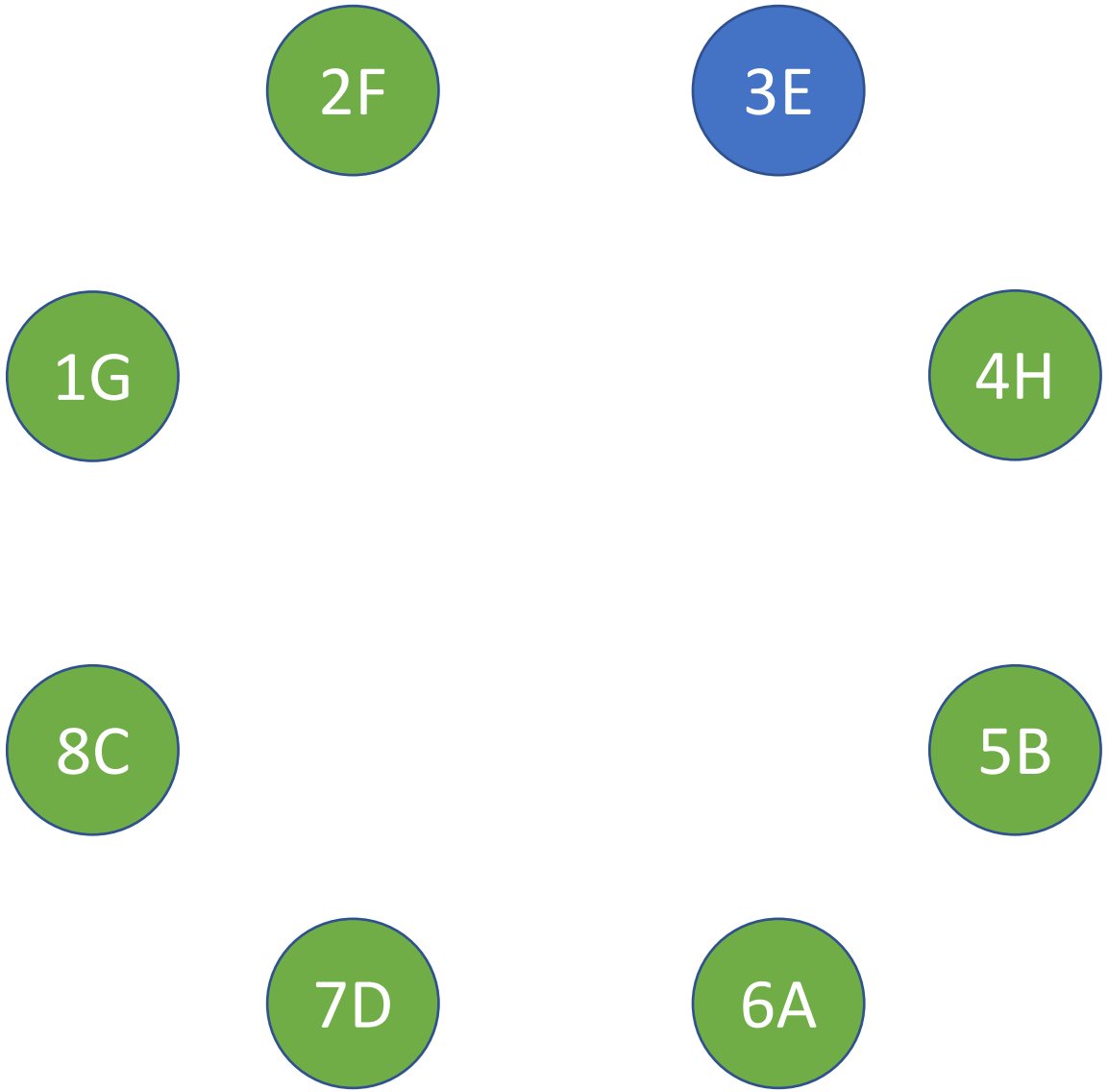
7D
6A


2F
3E
1G
4H
8C
5B
7D
6A

Final Allocation

1	2	3	4	5	6	7	8
D	B	G	H	B	A	D	C
G	D	F	D	C	E	F	E
B	C	A	E	G	G	A	D
E	F	C	A	E	H	G	G
F	G	E	G	F	C	B	B
C	E	H	B	A	F	C	F
H	H	D	C	H	B	H	A
A	A	B	F	D	D	E	H

Top Trading Cycles:
Dynamic vs Direct Implementations

Importance of Precise Language

To determine whether an **allocation** is:

- Pareto efficient, must know agents' preferences.
- Individually rational, must know agents' preferences and endowments.
- Core, must know agents' preferences and endowments.

Can an allocation be truthful? **No.**

We need to know what *would have happened with different preferences.*

In other words, a **mechanism** can be truthful.

We say that a mechanism is {Pareto efficient, individually rational, truthful} if it always selects an allocation with these properties.

Session 2 Study Guide

Concepts

- Endowment
- Truthful
  (Strategy-Proof,
  Incentive Compatible)
- Individually Rational
- The Core

Algorithms

- Top Trading Cycles

Facts

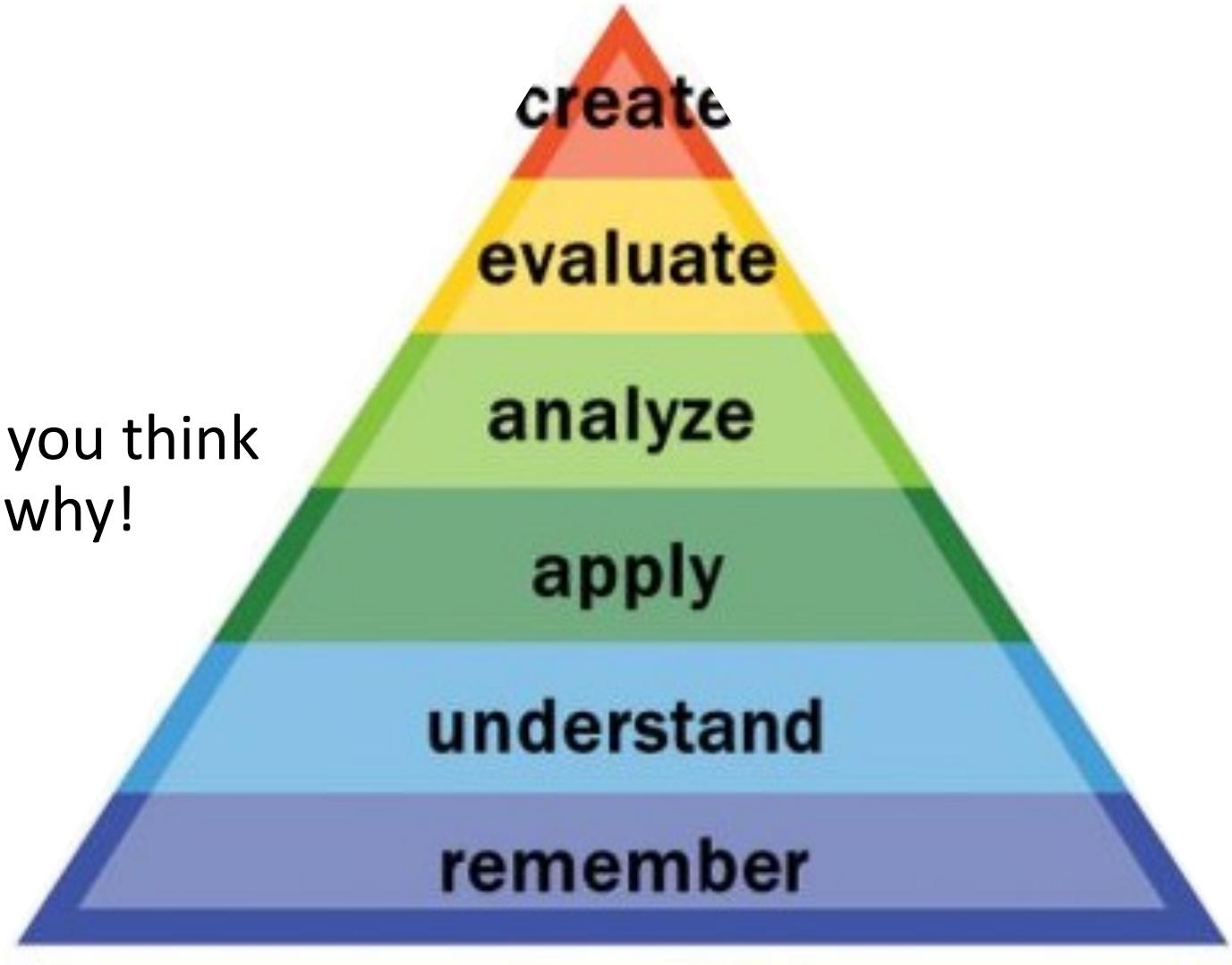
- Top Trading Cycles is PE, SP, IR.
- Top Trading Cycles gives the unique allocation in the core
- Top Trading Cycles is the **only** algorithm with these properties.

Questions? (On this topic or on HW.)

Answering Essay Questions

Tell me which properties you think are most important, and why!

Name an algorithm, and state its properties.



Two Settings

Designer controls all items

Algorithm: Serial Dictatorship

- Every order of agents is Pareto efficient and truthful.
- Can find any Pareto efficient allocation (with some order).

Could we use Top Trading Cycles?

Each item owned by an agent

Algorithm: Top Trading Cycles

- Pareto efficient, truthful, and individually rational.
- Finds the unique core allocation.

Could we use Serial Dictatorship?

Today’s Goal: Define Fairness

Intuitive definition: treat people equally!

This requires randomization.

Random Mechanisms

A **random mechanism** is a probability distribution over deterministic mechanisms.

- A random mechanism is **Pareto efficient** if all of the deterministic mechanisms which it might choose are Pareto efficient.
- A random mechanism is **truthful** (strategy-proof) if all of the deterministic mechanisms which it might choose are truthful.

There are other ways to define efficiency and truthfulness for random mechanisms, but today we will focus on the definitions above.

Top Trading Cycles from Random Endowments

1. Start from a uniformly random allocation.
2. Apply top trading cycles.

Random Serial Dictatorship

1. Place people in a uniform random order.
2. Apply serial dictatorship.

RSD vs TTC-RE

Both RSD and TTC-RE are Pareto efficient and truthful.

1. Do you think that Random Serial Dictatorship is fair?
   Does it treat people equally?

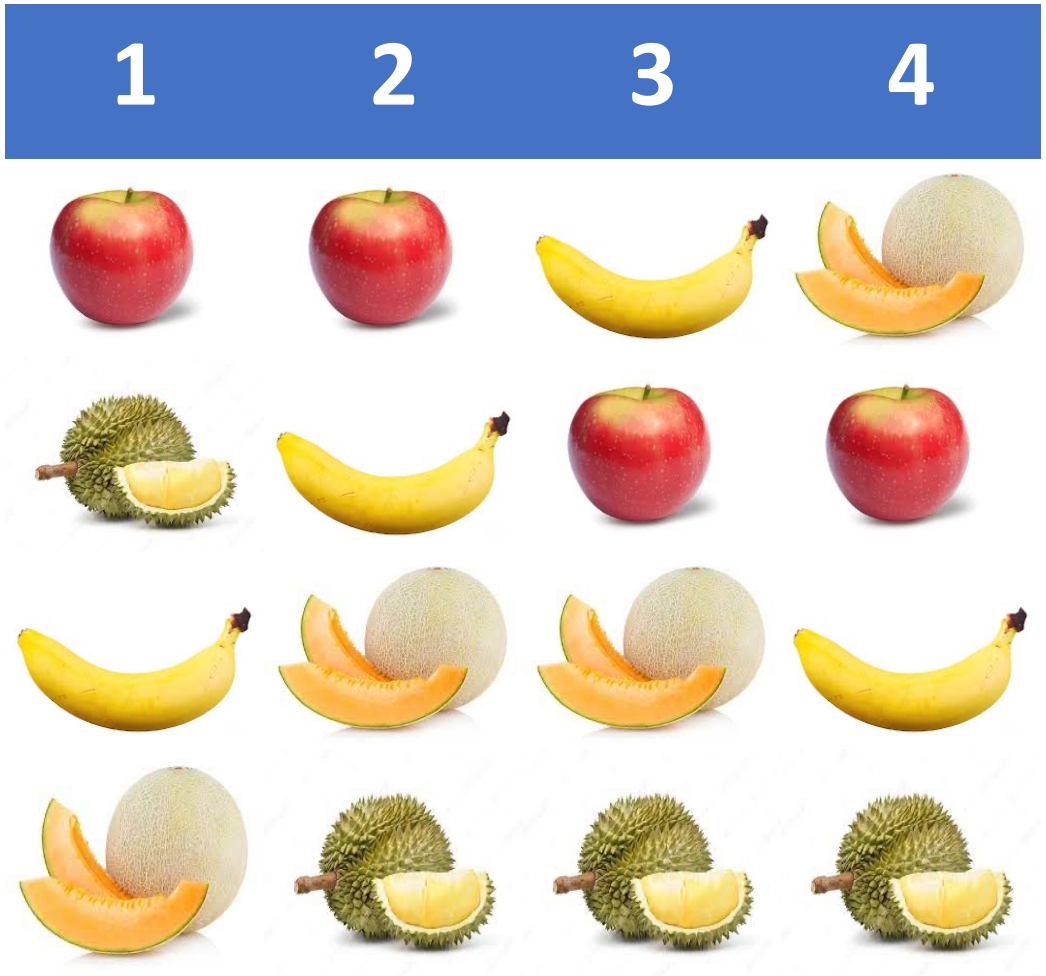
2. Do you think that Top Trading Cycles from Random Endowments is fair? Does it treat people equally?

3. Which algorithm would you recommend to policymakers?

RSD vs TTC-RE (Example 1)

Group Work:

1. If we use RSD:
   - What are the possible final assignments?
   - What is the probability of each?
2. If we use TTC-RE:
   - What are the possible final assignments?
   - What is the probability of each?

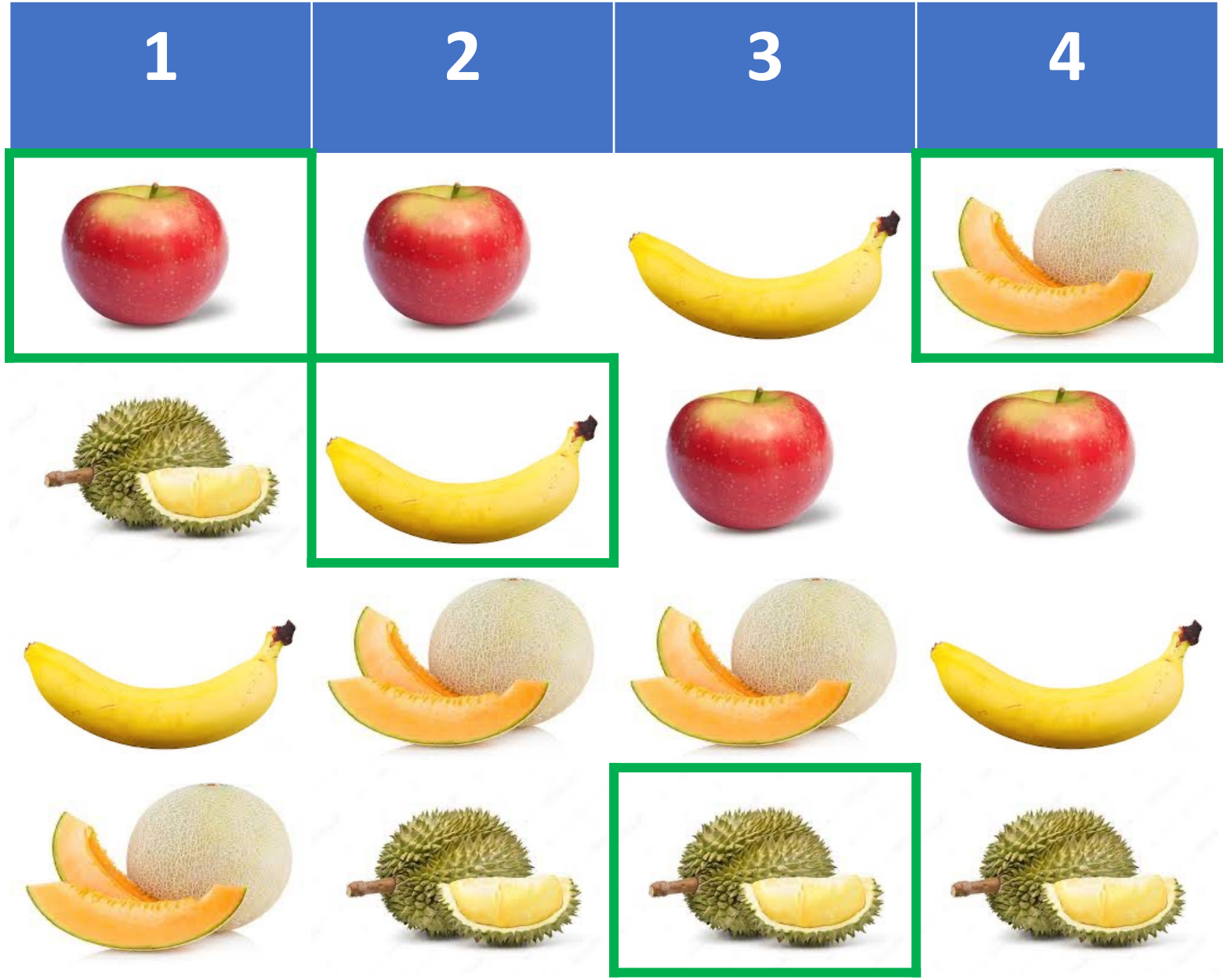


1	2	3	4
			
			
			
			

Under RSD, occurs with probability 1/2 (when 2 goes before 1).

1	2	3	4
			
			
			
			

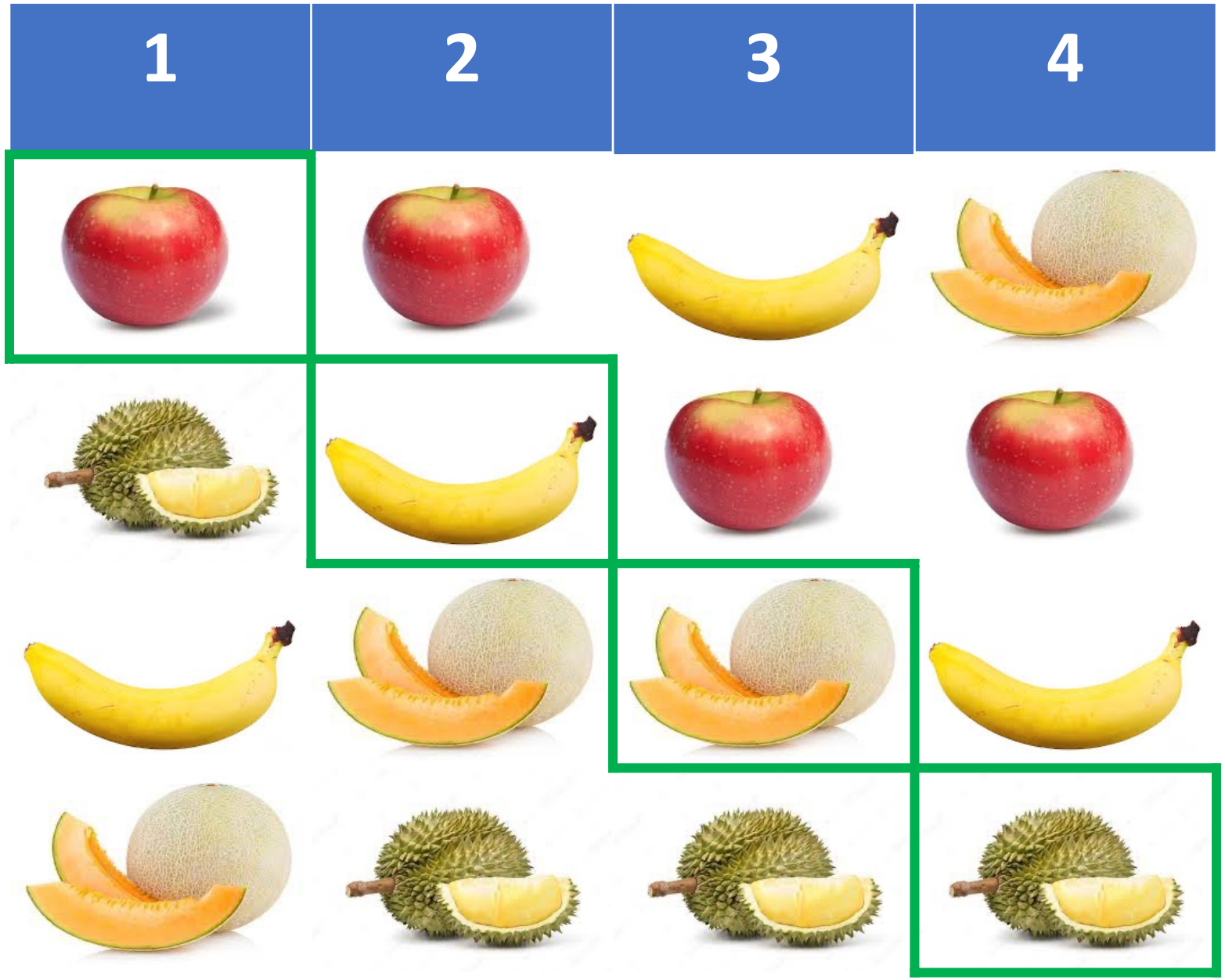
Under RSD, happens with probability 1/4 (when 2 goes last).



Under RSD, happens with probability 3/24 (orders 1423, 4123, 1243).

1	2	3	4
			
			
			
			

Under RSD, happens with probability 2/24 (orders 1324, 3124).

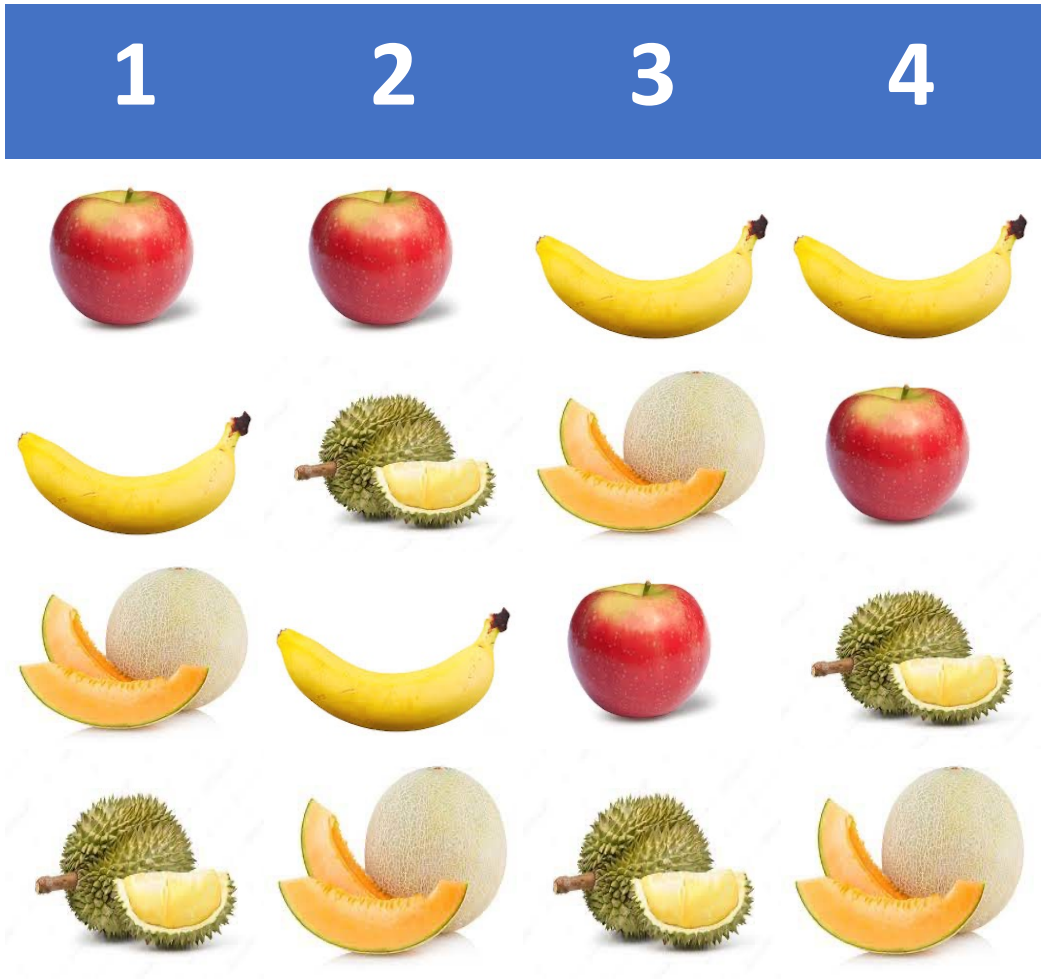


Under RSD, happens with probability 1/24 (when order is 1234).

RSD vs TTC-RE (Example 2)

Group Work:

1. If we use RSD:
   - What are the possible final assignments?
   - What is the probability of each?
2. If we use TTC-RE:
   - What are the possible final assignments?
   - What is the probability of each?



Equivalence of Mechanisms

Two random mechanisms **M** and **M'** are **equivalent** if for every preference profile **P** and allocation **A**, Prob(**M**(**P**) = **A**) = Prob(**M'**(**P**) = **A**).

Amazing fact: RSD and TTC-RE are equivalent!

(Proven by Knuth, Abdulkadiroglu and Sonmez)

How To Define Fairness?

"Treat People Equally" Could Mean:

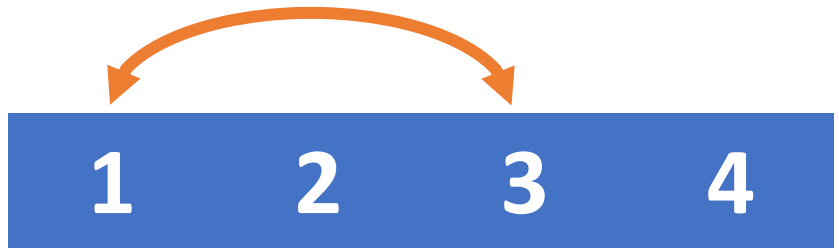
- **Equal Allocation.** People have the same chance of receiving each item
- **Equal Rank Distribution.** People have identical chance of receiving their first, second, third choice…
- **Symmetry.** (Defined on next slides.)

Other Definitions of Fairness (not discussed today):

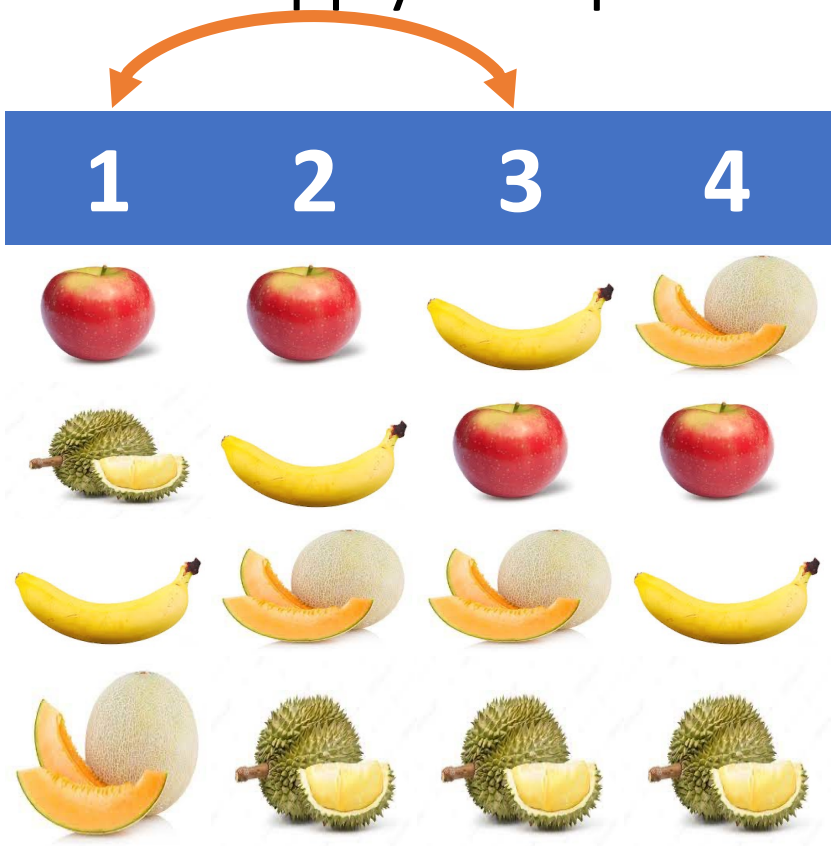
- Equal treatment of equals
- Envy-Free

Transposing Preferences

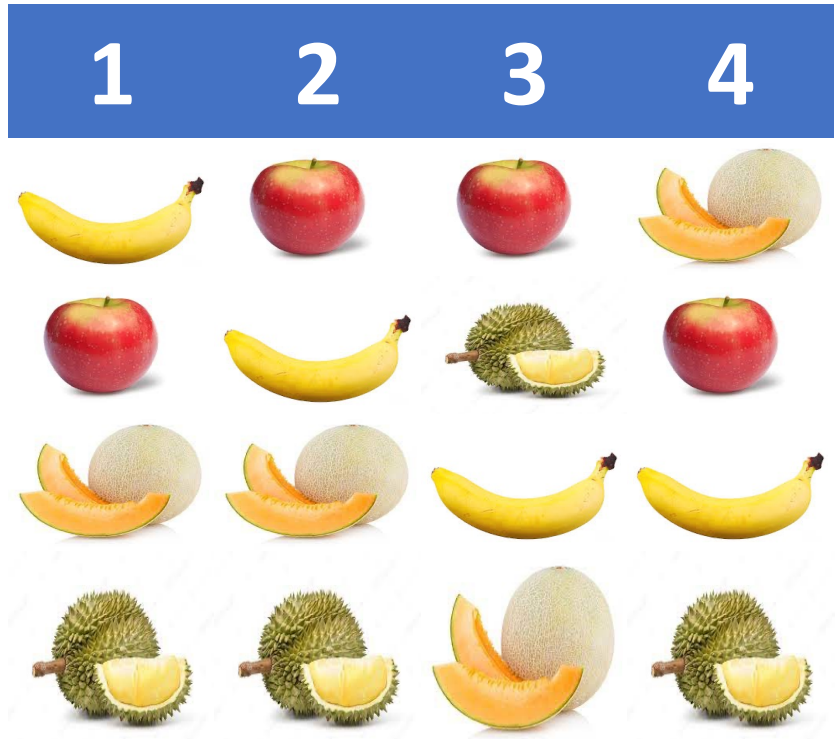
A **transposition** is a swap of two positions.



We can apply transposition **T** to any preference profile **P**:

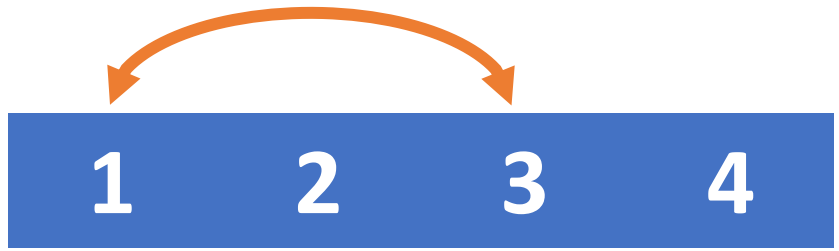


P ⇒ **T(P)**

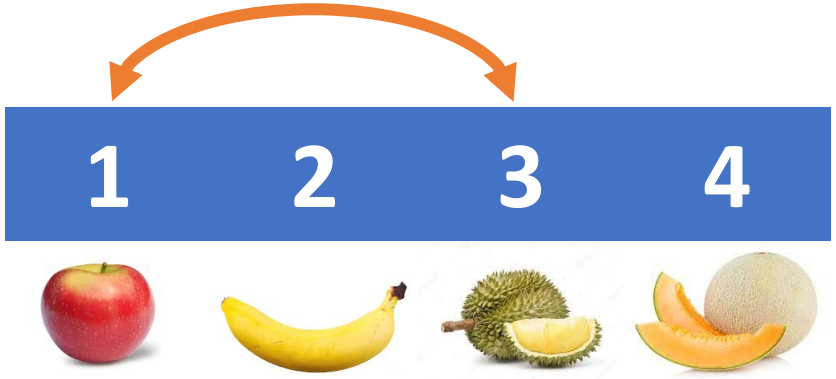


Transposing Allocations

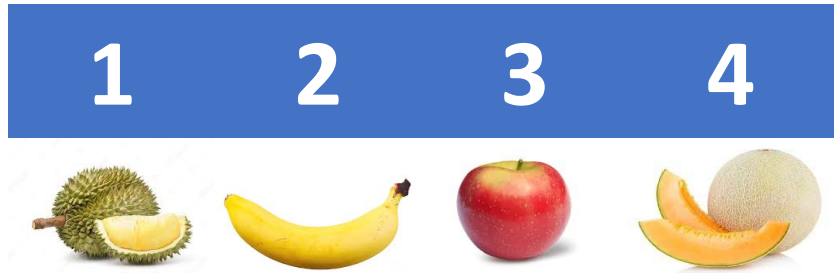
A **transposition** is a swap of two positions.



We can apply transposition T to any allocation A:



A ⇒ T(A)



Defining Symmetry

A random mechanism **M** is **symmetric** if for *any*
Preference profile **P**, Allocation **A**, and Transposition **T**:

$$\text{Prob}(\mathbf{M}(P) = A) = \text{Prob}(\mathbf{M}(T(P)) = T(A)).$$

Intuition:
We are not systematically favoring one agent over another.
Swapping two agent's preferences should just swap the final allocation.

Note: An equivalent definition says that this must hold for any permutation (possibly involving more than two agents).

How to generate symmetric mechanisms?

One Approach: Create deterministic mechanisms with different “roles”, and assign agents randomly to roles.

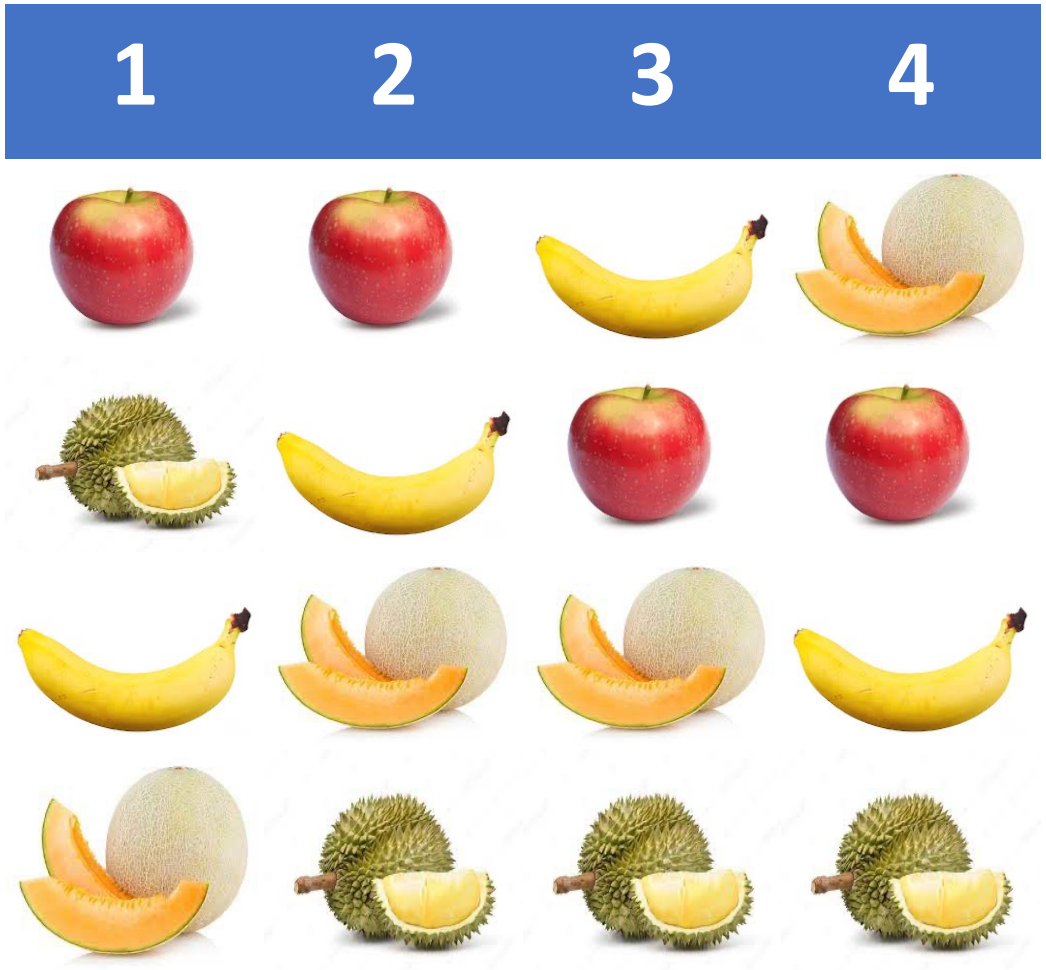
A New Mechanism

1. Give Apple, Banana, and Durian to Role 1.
2. Give the Cantaloupe to Role 4.
3. Roles 1 and 4 play TTC.
4. Once Role 1 matches, pass their excess fruit to Role 2. Have Roles 2 and 4 play TTC.
5. Once Role 2 matches, pass their excess fruit to Role 3. Have Roles 3 and 4 play TTC.

Group Work:

If we assign agents randomly to roles

- What are the possible final assignments?
- What is the probability of each?



Interesting Question

Is any mechanism which is **Pareto Efficient, Strategy-Proof,** and **Symmetric** equivalent to Random Serial Dictatorship?

We don't know!

In only three classes, we have gotten to the frontiers of human knowledge!

Closest Results:

1. If there are 3 agents and objects, 'Yes!' (Bogomolnaia and Moulin).
2. Any mechanism which is **Pareto Efficient, Symmetric*, Strategy-Proof,** and **Non-Bossy** is equivalent to RSD (Bade).
3. Any mechanism which is **Pareto Efficient, Symmetric,** and **Obviously Strategy-Proof** is equivalent to RSD (Pycia and Troyan).

Summary

All known mechanisms which are Pareto efficient, strategy-proof and symmetric are equivalent to Random Serial Dictatorship!

Despite (and sometimes because of) its simplicity, Random Serial Dictatorship is a reasonable solution in many cases.

Session 3 Study Guide

Concepts

- Random Mechanism
- Transposition
- Symmetry

Algorithms

- TTC-RE
- RSD

Facts

- RSD is Pareto efficient, truthful and symmetric.
- TTC-RE Pareto efficient, truthful and symmetric.
- Every known Pareto efficient, truthful, and symmetric mechanism is equivalent to RSD!

Reflection

1. Without looking at notes, define as many terms from class as you can.

2. Name 2-3 ideas or concepts to remember from this unit.

3. What questions do you have?

Coming Up: Intro to School Choice

- Concept Check
- Reflection and Critical Thinking
- Short Reading