Engineering Systems for Allocating Public Goods

Lesson 5: Deferred Acceptance

Response to Feedback

Plan for Today

1. School Choice Recap

2. Introduce “Deferred Acceptance” algorithm for finding assignments that are non-wasteful and respect priorities (“stable”).

3. Discuss the use of this algorithm for residency matching.

An Engineering Approach

First, specify criteria (“axioms”).

Then ask, which systems satisfy these criteria?

Unit 1 Recap

	Efficient	Fair (Symmetric)	Truthful
Random allocation	✗	✓	✓
Rank Efficient (w/random tie-break)	✓	✓	✗
Serial Dictatorship	✓	✗	✓
Random Serial Dictatorship (= TTC from Random Endowments)	✓	✓	✓

School Choice Recap

School choice algorithms typically don’t treat students equally
(are not symmetric).

Instead, each school has a priority ranking of students.

No Justified Envy (“Respects Priorities”):
Whenever one student envies another, the second student has higher priority at the coveted school.

Algorithm Review

	Pareto Efficient	Respects Priorities	Truthful
First Preferences First (Boston)	✓	✗	✗
Generalized Top Trading Cycles	✓	✗	✓

No algorithm guarantees Pareto Efficiency and Respecting Priorities

1	2	3
A	B	A
B	A	B
C	C	C

A	B	C
2	1	1
1	3	2
3	2	3

4 Pareto Efficient Allocations

None of them Respect Priorities!

Non-wastefulness and Stability

An assignment is **non-wasteful** if no student prefers an under-enrolled school to their own assignment.

Student Preferences

1	2	3
A	B	A
B	A	B
C	C	C

School Priorities

A	B	C
2	1	1
1	3	2
3	2	3

From Homework:
In this example, exactly one non-wasteful assignment respects priorities.

Questions

1. How to find assignment that is non-wasteful and respects priorities?

2. Could there be multiple assignments with these properties? If so, can we choose the “best” one?

3. Can we make this mechanism truthful?

Questions

1. How to find assignment that is non-wasteful and respects priorities?

   Deferred Acceptance

2. Could there be multiple assignments with these properties? If so, can we choose the "best" one?

   Yes!

   Student-proposing: best for students.

   School-proposing: best for schools.

3. Can we make this mechanism truthful?

   Next Class

In Class Example

Can you find an assignment that is non-wasteful and respects priorities?

1	2	3	4	5	6	7	8
A (62)	A (84)	B (60)	A (91)	B (86)	A (62)	A (94)	B (83)
B (67)	B (72)	A (90)	C (99)	A (75)	C (91)	C (89)	A (64)

9	10	11	12	13	14	15
A (86)	C (89)	B (70)	A (92)	B (78)	B (76)	B (84)
B (79)	B (75)	C (75)	B (86)	C (67)	A (87)	C (62)

A	B	C
7 (94)	12 (88)	4 (99)
12 (92)	5 (86)	6 (91)
4 (91)	15 (84)	7 (89)
3 (90)	8 (83)	10 (87)
14 (87)	9 (79)	11 (75)
9 (86)	13 (78)	13 (67)
2 (84)	14 (76)	15 (62)
5 (75)	10 (75)	1
8 (64)	2 (72)	2
6 (62)	11 (70)	3
1 (61)	1 (67)	5
10	3 (60)	8
11	4	9
13	6	12
15	7	14

A Dynamic Admissions Mechanism

1. Each school admits # applicants equal to vacancies (wait-lists others).
2. Students reject all but their favorite offer.

Repeat until each school either

i. is filled, or

ii. has admitted every applicant.

The direct version of this mechanism is called
“School Proposing Deferred Acceptance (DA)”

Schools make offers.

All student acceptances tentative (“deferred”) until the end.

In Class Example

What is the outcome of School-Proposing DA?

A: 7, 12, 4, 3, 14
B: 5, 15, 8, 9, 13
C: 6, 10, 11
Ø: 1, 2

1	2	3	4	5	6	7	8
A (62)	**A (84)**	**B (60)**	**A (91)**	**B (86)**	**A (62)**	**A (94)**	**B (83)**
B (67)	**B (72)**	**A (90)**	**C (99)**	**A (75)**	**C (91)**	**C (89)**	**A (64)**

9	10	11	12	13	14	15
A (86)	**C (89)**	**B (70)**	**A (92)**	**B (78)**	**B (76)**	**B (84)**
B (79)	**B (75)**	**C (75)**	**B (86)**	**C (67)**	**A (87)**	**C (62)**

A	B	C
7 (94)	12 (88)	4 (99)
12 (92)	5 (86)	6 (91)
4 (91)	15 (84)	7 (89)
3 (90)	8 (83)	10 (87)
14 (87)	9 (79)	11 (75)
9 (86)	13 (78)	13 (67)
2 (84)	14 (76)	15 (62)
5 (75)	10 (75)	1
8 (64)	2 (72)	2
6 (62)	11 (70)	3
1 (61)	1 (67)	5
10	3 (60)	8
11	4	9
13	6	12
15	7	14

Analyzing School Proposing DA

Why is this assignment non-wasteful?

Any school with vacancies has admitted every applicant.

Applicants choose their favorite school that admits them.

Why does this assignment respect priorities?

If you envy another student, you were not admitted to their school.

Schools admit from the top of their priority list.

Is there easy “verification” that this assignment respects priorities?

Publish the cutoff scores!

Could there be more than one non-wasteful allocation that respects priorities?

1	2
A	B
B	A

A	B
2	1
1	2

1A, 2B respects priorities (no envy).

1B, 2A respects priorities (no justified envy).

Student-Proposing Deferred Acceptance

1. Unassigned students apply to favorite school which they have not yet applied to.
2. Schools tentatively hold best applicants up to their capacity, reject the rest.

Repeat until each student either

i. is assigned,

ii. has applied to every school on his or her list.

"Student Proposing Deferred Acceptance (DA)"

Students make offers.

All school acceptances tentative ("deferred") until the end.

In Class Example

What is the outcome of Student-Proposing DA?

A: 7, 12, 4, 3, 9

B: 5, 15, 8, 13, 14

C: 6, 10, 11

∅: 1, 2

1	2	3	4	5	6	7	8
A (62)	**A (84)**	**B (60)**	**A (91)**	**B (86)**	**A (62)**	**A (94)**	**B (83)**
B (67)	**B (72)**	**A (90)**	**C (99)**	**A (75)**	**C (91)**	**C (89)**	**A (64)**

9	10	11	12	13	14	15
A (86)	**C (89)**	**B (70)**	**A (92)**	**B (78)**	**B (76)**	**B (84)**
B (79)	**B (75)**	**C (75)**	**B (86)**	**C (67)**	**A (87)**	**C (62)**

A	B	C
7 (94)	12 (88)	4 (99)
12 (92)	5 (86)	6 (91)
4 (91)	15 (84)	7 (89)
3 (90)	8 (83)	10 (87)
14 (87)	9 (79)	11 (75)
9 (86)	13 (78)	13 (67)
2 (84)	14 (76)	15 (62)
5 (75)	10 (75)	1
8 (64)	2 (72)	2
6 (62)	11 (70)	3
1 (61)	1 (67)	5
10	3 (60)	8
11	4	9
13	6	12
15	7	14

Break

Residency Matching

Problem: Market was "unraveling." Hospitals raced to make earlier and earlier offers. (Years before graduation!)

Solution: A centralized match in final year of medical school. Students and hospitals rank each other. Then apply hospital-proposing DA.

Timeline:

- Reform happened in 1951
- Earliest academic study of DA in 1962 (Gale & Shapley)
- Nobody realized the connection until 1984! (Roth)

Stability

What does this remind you of?

The CORE!

Concern: We recommend a match, but cannot enforce it.

Will anybody be able to profitably deviate from recommendation?

An assignment is **individually rational** if each matched doctor and hospital agree that being matched to each other is better than being unmatched.

A doctor *d* and a hospital *h* form a **blocking pair** if

(i) *d* prefers *h* to *d*'s suggested assignment, and

(ii) *h* has an unfilled position OR prefers *d* to another doctor assigned to it.

An assignment is **stable** if it is individually rational and has no blocking pairs.

One Definition, Two Interpretations

Fact: An assignment is stable if and only if it is non-wasteful and respects priorities.

	Residency Matching	School Choice
Terminology	Hospital Preferences	School Priorities
	Stable	Nonwasteful + Respects Priorities
Motivation	People might deviate	Procedural Fairness

The Core: Comparing to Unit 1

Both Settings:

- Core = Outcomes such that no subset can agree on profitable deviation.

Allocation with endowments:

- No direct relationship between what I get and who my object goes to.
- Unique core allocation.

Two-sided matching:

- Each agent is also an object!
- Core (stable matchings) **not unique**.

Which Stable Matching Should We Choose?

School-Proposing

A	B	C
7 (94)	12 (88)	4 (99)
12 (92)	5 (86)	6 (91)
4 (91)	15 (84)	7 (89)
3 (90)	8 (83)	10 (87)
14 (87)	**9 (79)**	11 (75)
9 (86)	13 (78)	13 (67)
2 (84)	14 (76)	15 (62)
5 (75)	10 (75)	1
8 (64)	2 (72)	2
6 (62)	11 (70)	3

Student-Proposing

A	**B**	**C**
7 (94)	12 (88)	4 (99)
12 (92)	5 (86)	6 (91)
4 (91)	15 (84)	7 (89)
3 (90)	8 (83)	10 (87)
14 (87)	9 (79)	11 (75)
9 (86)	13 (78)	13 (67)
2 (84)	14 (76)	15 (62)
5 (75)	10 (75)	1
8 (64)	2 (72)	2
6 (62)	11 (70)	3

1. Which students are unassigned in each case?
2. Which students prefer
   School-Proposing?
   Student-Proposing?
3. Which students are indifferent?

Amazing Facts

1. **Rural Hospital Theorem**: any two stable assignments assign the same students and the same number of seats at each hospital!

2. **Student Optimality:** all students agree that student-proposing DA gives the *best possible* stable match, and hospital-proposing DA gives the *worst possible* stable match!

Deferred Acceptance Practice

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Group Work:

1. What is the outcome of Student-Proposing Deferred Acceptance?
2. What is the outcome of School-Proposing Deferred Acceptance?
3. Under each mechanism, can any student benefit from misreporting?

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 1 applies to B, 2, 3, 4 apply to C.
Students 2 and 4 are rejected by C.

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 2 applies to B, student 4 applies to D.
Student 2 is rejected by B.

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 2 applies to D.
Student 4 is rejected by D.

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 4 applies to B.
Student 1 is rejected by B.

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 1 applies to D.
Student 2 is rejected by D.

Deferred Acceptance: Student Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

Student 2 applies to A.

No further rejections, so the match DACB is finalized.

Deferred Acceptance: School Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
~~3~~	3	~~1~~	1
1	4	3	2
4	1	2	4
2	2	4	3

Schools A and B admit student 3, schools C and D admit student 1.
School A is rejected by student 3, school C is rejected by student 1.

Deferred Acceptance: School Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
~~3~~	~~3~~	~~1~~	1
~~1~~	4	3	2
4	1	2	4
2	2	4	3

School A admits student 1, school C admits student 3.
School A rejected by student 1, school B rejected by student 3.

Deferred Acceptance: School Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
~~3~~	~~3~~	~~1~~	1
~~1~~	4	3	2
~~4~~	1	2	4
2	2	4	3

Schools A and B admit student 4.

School A rejected by student 4.

Deferred Acceptance: School Proposing

Student Preferences

1	2	3	4
B	C	C	C
D	B	B	D
A	D	D	B
C	A	A	A

School Priorities

A	B	C	D
3	3	1	1
1	4	3	2
4	1	2	4
2	2	4	3

School A admits student 2.

No further rejections, so the match DACB is finalized.

Summary

	Pareto Efficient	Respects Priorities	Truthful
First Preferences First (Boston)	✓	✗	✗
Generalized Top Trading Cycles	✓	✗	✓
School Proposing Deferred Acceptance	✗	✓	?
Student Proposing Deferred Acceptance	✗	✓	?

No algorithm guarantees Pareto Efficiency and Respecting Priorities (last class)

Study Guide

Concepts

- Unraveling
- Blocking Pair
- Stability
- The Core (new context)

Algorithms

- School-Proposing Deferred Acceptance
- Student-Proposing Deferred Acceptance

Facts

- Stability = Non-wasteful + No Justified Envy
- There may be multiple stable assignments.
- Stable assignments can be verified by “cutoff scores”
- Rural Hospital Theorem: any two stable assignments assign the same students and the same number of seats at each school.
- Student-Proposing DA finds “student optimal” stable assignment: all students agree it’s at least as good as any other stable assignment.
- School-Proposing DA finds “student pessimal” and “school optimal” stable assignment.

Graduate Housing, Revisited (Homework)

What if people have preferences over **roommates**?

Setting:

- We have 2-bedroom apartments, must pair people up.
- Same as classic stable matching problem, except that any two people can be paired (not two-sided).
- Does a stable matching always exist? How can we find it?