# Engineering Systems for Allocating Public Goods

Stable Matching in Practice:

Dealing with Couples and Other Complexities

# Plan for Today

1. Dynamic Deferred Acceptance for Clinical Psychology
2. NRMP Redesign: Match Complexities
3. NRMP Redesign: Which Side Proposes?
4. Recent theory inspired by these stories
5. Which other markets could or should be centralized?

# Stable Matching Recap

In “simple” many-to-one matching markets,

1. Stable matchings always exist.
2. The set of assigned students and assigned positions is the same for every stable matching.
3. There is a student-optimal stable match.
4. Student-proposing DA (with any proposal order) finds the student-optimal match.
5. Student-proposing DA is truthful for students.
6. Student-proposing DA is population monotonic.

# Clinical Psychology Match

5. An applicant must respond immediately to each offer tendered in one of three ways. The offer may be accepted, rejected or "held."
   a. *Accepting* the offer constitutes a binding agreement between applicant and internship program.
   b. *Refusing* the offer terminates all obligations on either side and frees the internship program to offer the position to another applicant.
   c. *Holding* the offer means that the offer remains valid until the applicant notifies the program of rejection or acceptance, or until the end of selection day.
6. Applicants may "HOLD" no more than one active offer at a time.
   a. If an applicant is holding an offer from one program and receives an offer from a more preferred program, s/he may accept or "hold" the second offer provided that the less preferred program is notified *immediately* that the applicant is rejecting the previously held offer.
   b. If a program confirms that an applicant is holding more than one offer, the program is free to withdraw their previously tendered offer of acceptance, and to offer that position to another applicant *after* the offending applicant is notified of that decision.

"Offer Day" from 9 am to 4 pm.

Offers made over the phone.

With straightforward behavior and no deadline, equivalent to program-proposing Deferred Acceptance.

# What Actually Happened?

*On selection day the codirectors said that their general strategy was ‘‘don’t tie up offers with people who will hold them all day.’’ They therefore decided to make their first offers (for their five positions) to numbers 1, 2, 3, 5, and 12 on their rank-order list, with the rationale being that numbers 3, 5, and 12 had indicated that they would accept immediately and that 1 and 2 were so attractive as to be worth taking chances on.*

*Two phones were used to make these calls, starting precisely at 9:00 a.m. central standard time. Candidates 3, 5, and 12 accepted immediately, as promised. Candidate 1 was reached at 9:05 (on the fourth attempt, after three busy signals) and held the offer until 9:13, when he called back to reject it.*

Roth and Xing, 1997

*During this period, an incoming call (on a third phone whose number had been given to candidates) was received from the candidate ranked eighth, who now said that the program was her first choice. She was thanked and told she was still under consideration, and when candidate 1 called to reject the offer he was holding, the codirectors decided to make the next offer to candidate 8 (and not to number 4, as initially planned).*

*The offer to number 8 was then made and accepted immediately, and while that phone call was in progress, an incoming call from candidate 2 informed them that she had accepted another position. The decision was then made to offer the remaining position next to the highest-ranked remaining candidate who had indicated that he would accept immediately, number 10, and this offer was accepted at 9:21.*

*After the briefest of celebrations, the codirectors called the remaining candidates to inform them that all positions were filled. These calls were completed by 9:35, 35 minutes after the opening of the market. The five positions were filled with the candidates initially ranked 3, 5, 8, 10, and 12.*

Roth and Xing, 1997

# Simulation Results

TABLE 1

MEDICAL MODEL TELEPHONE MARKET: RESULTS OF 100 SIMULATIONS FOR EACH OF THREE TURNAROUND TIMES

| | NUMBER OF MINUTES REQUIRED TO MAKE AN OFFER (and Reject One) | | |
|---|---|---|---|
| | 5<br>1 | 10<br>2 | 25<br>5 |
| | A. Preferences over 20 Firms; Uncorrelated Random Preferences | | |
| Mean time to termination at a stable outcome | 18:18<br>(8:10) | 36:32<br>(16:20) | 91:14<br>(40:52) |
| Median time to termination | 16:24 | 32:39 | 81:19 |
| Mean time by which 90% of students have received an offer | 1:02 | 2:03 | 5:04 |
| Mean time by which 99% of students have received an offer | 5:19 | 10:35 | 26:22 |
| Longest time to termination | 39:25 | 78:25 | 196:22 |
| Shortest time to termination | 4:59 | 9:55 | 25:00 |
| | B. Preferences over All 50 Firms; Uncorrelated Random Preferences | | |
| Mean time to termination at a stable outcome | 22:53<br>(12:03) | 45:35<br>(24:04) | 113:42<br>(60:12) |
| Median time to termination | 18:57 | 37:44 | 94:09 |
| Mean time by which 90% of students have received an offer | 1:09 | 2:15 | 5:35 |
| Mean time by which 99% of students have received an offer | 7:02 | 13:55 | 34:39 |
| Longest time to termination | 55:15 | 110:03 | 274:48 |
| Shortest time to termination | 6:10 | 12:12 | 30:50 |

NOTE.—Standard deviations are in parentheses.

TABLE 2

HOURLY PROGRESS OF THE MEDICAL MODEL TELEPHONE MARKET (100 Simulations)

| Hour | Number of Students with at Least One Offer | Number of Students with an Offer from the Firm They Will Ultimately Match With | Number of Offers Made | Number of Offers Not Rejected Immediately |
|---|---|---|---|---|
| 0 | .00 | .00 | .00 | .00 |
| 1 | 178.47 | 86.32 | 400.08 | 278.06 |
| 2 | 191.24 | 116.06 | 531.96 | 333.90 |
| 3 | 194.83 | 132.75 | 602.36 | 360.04 |
| 4 | 196.50 | 143.81 | 648.58 | 375.70 |
| 5 | 197.41 | 152.14 | 681.79 | 386.80 |
| 6 | 198.02 | 158.48 | 707.38 | 395.01 |
| 7 | 198.37 | 163.37 | 727.89 | 401.10 |
| 8 | 198.54 | 167.66 | 745.23 | 406.29 |
| 9 | 198.68 | 171.46 | 761.06 | 410.70 |
| 10 | 198.84 | 174.77 | 775.07 | 414.65 |
| 11 | 198.97 | 177.59 | 787.29 | 417.85 |
| 12 | 199.05 | 180.32 | 798.49 | 421.03 |
| 13 | 199.18 | 182.78 | 808.49 | 423.75 |
| 14 | 199.29 | 184.76 | 817.30 | 425.99 |
| 15 | 199.41 | 186.72 | 824.77 | 428.12 |
| 16 | 199.44 | 188.26 | 831.41 | 429.84 |
| 17 | 199.51 | 189.75 | 837.30 | 431.51 |
| 18 | 199.57 | 191.04 | 842.61 | 432.89 |
| 19 | 199.62 | 192.19 | 847.21 | 434.17 |
| 20 | 199.67 | 193.11 | 851.38 | 435.20 |
| 21 | 199.69 | 193.91 | 854.99 | 436.09 |
| 22 | 199.71 | 194.70 | 858.47 | 436.96 |
| 23 | 199.76 | 195.47 | 861.63 | 437.78 |
| 24 | 199.77 | 195.98 | 864.35 | 438.32 |

# What if preferences are correlated?

TABLE 6

MEDICAL AND PSYCHOLOGY MARKET SIMULATIONS: VARYING THE CORRELATION OF PREFERENCES
(Students Have Preferences over All 50 Firms; 100 Simulations)

| | PREFERENCES | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Case 1: Students Have Uncorrelated Random Preferences; Firms Have Uncorrelated Random Preferences | | Case 2: Students Have Uncorrelated Random Preferences; Firms Have Identical Preferences | | Case 3: Students Have Identical Preferences; Firms Have Identical Preferences | | Case 4: Students Have Identical Preferences; Firms Have Uncorrelated Random Preferences | |
| | Medical Market | Psychology Market | Medical Market | Psychology Market | Medical Market | Psychology Market | Medical Market | Psychology Market |
| Mean* time to termination | 22:53 (12:03) | 8:39 (:48) | 25:09 (:45) | 18:10 (:14) | 20:46 (:18) | 17:12 (:17) | 13:16 (2:18) | 8:29 (:32) |
| Mean time by which 90% of students have received an offer | 1:09 | 1:09 | 22:06 | 16:10 | 18:51 | 15:12 | 1:18 | 1:18 |
| Mean time by which 99% of students have received an offer | 7:02 | 6:21 | 24:50 | 17:57 | 20:36 | 16:58 | 7:53 | 6:52 |
| Mean number of blocking firms | 0 | 2.23 (.85) | 0 | 47.75 | 0 | 37.1 (2.05) | 0 | .68 (.68) |
| Mean number of blocking students | 0 | 31 (12.83) | 0 | 151.31 (3.71) | 0 | 156.13 (7.48) | 0 | 1.72 (2.23) |

NOTE.—Standard deviations are in parentheses.
* The corresponding medians are very close to the means.

# Dynamic vs Direct Implementations

- Dynamic process took too long.
- Additional strategies available in dynamic implementation.
- Harder to say ‘no’ (or hold an offer) over the phone.

# Break

# The NRMP Redesign

# Incorporating Couples

Informal definition of blocking pair:

A matching is **blocked** by a coalition if the coalition can agree upon an allocation that all members of the coalition prefer.

# Incorporating Couples

| A | B |
|---|---|
| 4 | 2 |
| 1 | 3 |

There are two hospitals, A and B, each with a single position.

There are four doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| A | B | B | A |

Student-Proposing DA:

1. Doctors 1 and 2 apply to A and B, respectively.
2. Doctor 3 applies to B, and is rejected.
3. Doctor 4 applies to A, causing Doctors 1 and 2 to be rejected.

Final assignment: Ø Ø Ø A.

Unstable! B wants to hire 3.

May fail to find a stable assignment!

# Incorporating Couples

| A | B |
|---|---|
| 4 | 2 |
| 1 | 3 |

There are two hospitals, A and B, each with a single position.

There are four doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| A | B | B | A |

Student-Proposing DA (reversed processing order):

1. Doctor 4 applies to A.
2. Doctor 3 applies to B.
3. Doctors 1 and 2 apply to A and B, and are rejected.

Final assignment: Ø Ø B A.

This is the only stable assignment.

Final outcome depends on processing order!

# Incorporating Couples

| A | B |
|---|---|
| 4 | 2 |
| 1 | 3 |

There are two hospitals, A and B, each with a single position.

There are four doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| A | B | B | Ø |

Student-Proposing DA (eliminate agent 4):

1. Doctor 3 applies to B.
2. Doctors 1 and 2 apply to A and B, and are accepted.

Final assignment: A B Ø Ø.

This is the **only** stable assignment.

Adding Doctor 4 helps Doctor 3!

No stable mechanism is population monotonic!

| A | B |
|---|---|
| 1 | 1 |
| 3 | 3 |
| 2 | 2 |

There are two hospitals, A and B, each with a single position.

There are three doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 |
|---|---|---|
| A | B | A |
| B | A | B |

There are four individually rational matchings:

A B Ø **blocked** by 3B

B A Ø **blocked** by 3A

Ø Ø B **blocked** by 3A or 1B 2A

Ø Ø A **blocked** by 1A, 2B

No stable matching exists!

| A | B |
|---|---|
| 1 | 1 |
| 3 | 3 |
| 2 | 2 |

There are two hospitals, A and B, each with a single position.

There are three doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 |
|---|---|---|
| A | B | A |
| B | A | B |

Student-Proposing DA (couple processed first):

1. 1 and 2 propose to A and B.
2. 3 proposes to A, and is rejected.
3. 3 proposes to B, causing 2 to be rejected.
4. 1 and 2 propose to B and A, causing 3 to be rejected.

Final match: B A Ø.

| A | B |
|---|---|
| 1 | 1 |
| 3 | 3 |
| 2 | 2 |

There are two hospitals, A and B, each with a single position.

There are three doctors. Doctors 1 and 2 are a couple.

| 1 | 2 | 3 |
|---|---|---|
| A | B | B |
| B | A | A |

Student-Proposing DA (couple processed first):

1. 1 and 2 propose to A and B.
2. 3 proposes to B, causing 2 to be rejected.
3. 1 and 2 propose to B and A, causing 3 to be rejected.
4. 3 proposes to A, causing 2 to be rejected.

Final match: Ø Ø A.

Student-proposing DA not truthful!

# Stable Matching Recap

With Couples

In “simple” many-to-one matching markets,

1. Stable matchings always exist. 
2. The set of assigned students and assigned positions is the same for every stable matching. 
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5. Student-proposing DA is truthful for students. 
6. Student-proposing DA is population monotonic. 

# Similar problems if programs want pairs of residents (even if no couples)

Definition of blocking?

2 hospitals, 2 doctors.

- Hospital A (2 positions): $\{1, 2\} \succ \emptyset$
- Hospital B (1 position): $1 \succ 2 \succ \emptyset$
- Doctor 1: $A \succ B$
- Doctor 2:

  If $\emptyset$, only stable match is B $\emptyset$.

  If $A \succ B$, only stable match is A A.

  If $B \succ A$, there is no stable match!

Fails population monotonicity.

(A A blocked by 2B, $\emptyset$ B blocked by 1B, B $\emptyset$ blocked by 12A)

# What to do?

# A Look at the Data

A stable match was always found.

TABLE 2—COMPARISON OF RESULTS BETWEEN ORIGINAL NRMP ALGORITHM AND APPLICANT-PROPOSING ALGORITHM

| Result | 1987 | 1993 | 1994 | 1995 | 1996 |
|---|---|---|---|---|---|
| Programs with new position(s) filled | 0 | 0 | 2 | 1 | 1 |
| Programs with new unfilled position(s) | 1 | 0 | 2 | 0 | 0 |

The rural hospital theorem almost held.

TABLE 4—UPPER LIMIT OF THE NUMBER OF APPLICANTS WHO COULD BENEFIT BY TRUNCATING THEIR LISTS AT ONE ABOVE THEIR ORIGINAL MATCH POINT

| | Upper limit | |
|---|---|---|
| Year | Preexisting NRMP algorithm | Applicant-proposing algorithm |
| 1987 | 12 | 0 |
| 1993 | 22 | 0 |
| 1994 | 13 | 2 |
| 1995 | 16 | 2 |
| 1996 | 11 | 9 |

Few applicants could benefit from misreporting.

TABLE A1—EFFECTS OF SEQUENCE IN WHICH PROGRAMS ARE PROCESSED

A. *Results with Programs Processed in Descending Code Order Compared to Original Results with Preexisting NRMP Algorithm*

| Result | 1993 | 1994 | 1995 |
|---|---|---|---|
| Programs | | | |
| Improve | none | 2 | 2 |
| Do worse | 2 | 2 | none |
| Applicants | | | |
| Improve | 2 | 2 | none |
| Do worse | none | 2 | 2 |

Sequencing order hardly mattered.

# A Helpful Analogy

*Consider the design of suspension bridges. The Newtonian physics they embody is beautiful both in mathematics and in steel, and college students can be taught to derive the curves that describe the shape of the supporting cables. But no bridge could be built based only on this elegant theoretical treatment, in which the only force is gravity, and all beams are perfectly rigid. Real bridges are built of steel and rest on rock and soil and water, and so bridge design also concerns metal fatigue, soil mechanics, and the forces of waves and wind.*

Roth and Peranson, 1999

*Many design questions concerning these real-world complications cannot be answered analytically but, instead, must be explored using physical or computational models. Often these involve estimating magnitudes of phenomena missing from the simple Newtonian model, some of which are small enough to be of little consequence, while others will cause the bridge to fall down if not adequately addressed. Just as no suspension bridges could be built without an understanding of the underlying physics, neither could any be built without understanding many additional features, also physical in nature, but more varied and complex than addressed by the simple model.*

Roth and Peranson, 1999

# Practical Problems Inspire New Theory

**Large Markets**

Kojima, Pathak, Roth, 2010

In large random markets with few couples stable matchings exist.

In a "large market" with couples, stable matchings exist.

(Assumptions needed!)

Azevedo and Hatfield, 2018

Che, Kim, and Kojima 2019

**Modifying Capacities**

Given any instance of a stable matching problem with couples, we can find a "nearby" instance in which a stable matching exists:

Nguyen and Vohra, 2018

- Same preferences
- Each hospital's capacity changed by at most 2.
- Total hospital capacity changed by at most 4.

# Which Side Proposes?

TABLE 2—COMPARISON OF RESULTS BETWEEN ORIGINAL NRMP ALGORITHM AND APPLICANT-PROPOSING ALGORITHM

| Result | 1987 | 1993 | 1994 | 1995 | 1996 |
|---|---|---|---|---|---|
| *Applicants:* | | | | | |
| Number of applicants affected | 20 | 16 | 20 | 14 | 21 |
| Applicant-proposing result preferred | 12 | 16 | 11 | 14 | 12 |
| Current NRMP result preferred | 8 | 0 | 9 | 0 | 9 |
| U.S. applicants affected | 17 | 9 | 17 | 12 | 18 |
| Independent applicants affected | 3 | 7 | 3 | 2 | 3 |
| Difference in result by rank number | | | | | |
| 1 rank | 12 | 11 | 13 | 8 | 8 |
| 2 ranks | 3 | 1 | 4 | 2 | 6 |
| 3 ranks | 2 | 3 | 2 | 2 | 3 |
| More than 3 ranks | 2 | 1 | 1 | 2 | 3 |
| | (max 9) | (max 4) | (max 5) | (max 6) | (max 6) |
| New matched | 0 | 0 | 0 | 0 | 1 |
| New unmatched | 1 | 0 | 0 | 0 | 0 |
| *Programs:* | | | | | |
| Number of programs affected | 20 | 15 | 23 | 15 | 19 |
| Applicant-proposing result preferred | 8 | 0 | 12 | 1 | 10 |
| Current NRMP result preferred | 12 | 15 | 11 | 14 | 9 |
| Difference in result by rank number | | | | | |
| 5 or fewer ranks | 5 | 3 | 9 | 6 | 3 |
| 6–10 ranks | 5 | 3 | 3 | 5 | 3 |
| 11–15 ranks | 0 | 5 | 1 | 3 | 1 |
| More than 15 ranks | 9 | 4 | 6 | 0 | 11 |
| | (max 178) | (max 36) | (max 31) | | (max 191) |
| Programs with new position(s) filled | 0 | 0 | 2 | 1 | 1 |
| Programs with new unfilled position(s) | 1 | 0 | 2 | 0 | 0 |

TABLE 3—DIFFERENCE IN RESULT WHEN ALGORITHM CHANGED FROM PREEXISTING SPECIALTY MATCH TO APPLICANT-PROPOSING

| Year | Difference |
|---|---|
| 1991 | none |
| 1992 | 2 applicants improve, 2 programs do worse |
| 1993 | 2 applicants improve, 2 programs do worse |
| 1994 | none |
| 1996 | none |

Doesn't really matter!

# Practical Problems Inspire New Theory

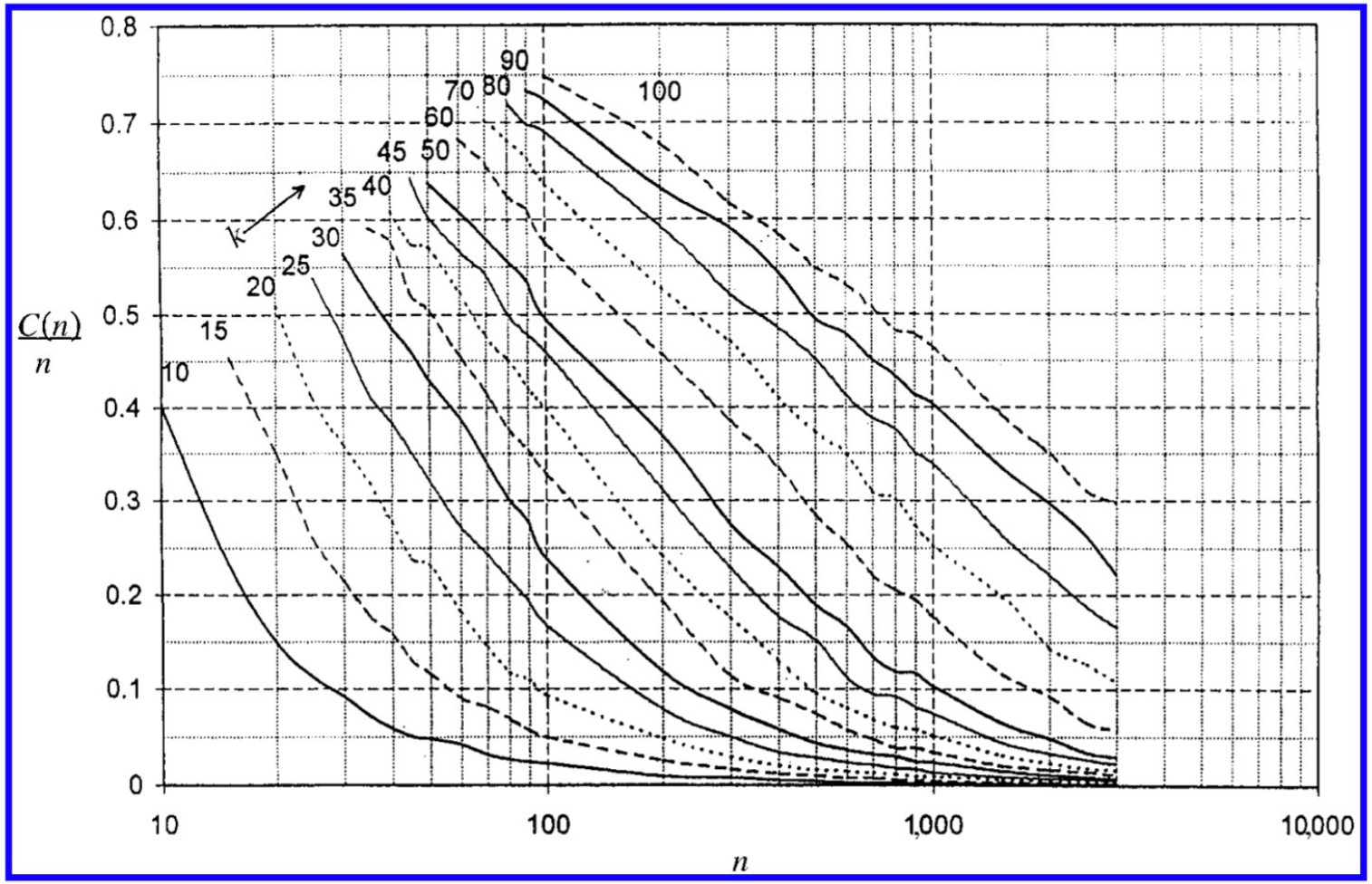
Why was there an “almost unique” stable match?

Theory: with n men, n women, and iid random preferences, the proposing side makes a big difference!

Explanations in Roth-Peranson:

- Short lists
- Correlated preferences

# Simulation Results: Incomplete Lists



# Summary

- Dynamic mechanisms introduce many more possibilities for strategic behavior than their direct counterparts.
- Real-world markets include many complexities, which may break nice theoretical properties.
- Nevertheless, simple theory can provide helpful guidance for these markets.
- In the medical residency match, the choice of proposing side had a minimal effect on the final assignment (99.9% of students have a unique stable match).

# Coming Up

Practical issue in school choice: how to break ties in priority?