Engineering Systems for Allocating Public Goods

Stable Matching in School Choice:

Dealing with Ties in Priority

Recap: Last Class

Real-world markets include many complexities, which may break nice theoretical properties.

Nevertheless, simple theory can provide helpful guidance for these markets.

In the medical residency match, the choice of proposing side had a minimal effect on the final assignment (99.9% of students have a unique stable match).

Plan for Today

1. Why and when is there a “nearly unique” stable matching?

2. Theory of Stable Matching with Indifferences

3. Breaking Ties With Lotteries: Evidence from Practice

Which Side Proposes?

TABLE 2—COMPARISON OF RESULTS BETWEEN ORIGINAL NRMP ALGORITHM AND APPLICANT-PROPOSING ALGORITHM

Result	1987	1993	1994	1995	1996
Applicants:					
Number of applicants affected	20	16	20	14	21
Applicant-proposing result preferred	12	16	11	14	12
Current NRMP result preferred	8	0	9	0	9
U.S. applicants affected	17	9	17	12	18
Independent applicants affected	3	7	3	2	3
Difference in result by rank number					
1 rank	12	11	13	8	8
2 ranks	3	1	4	2	6
3 ranks	2	3	2	2	3
More than 3 ranks	2	1	1	2	3
	(max 9)	(max 4)	(max 5)	(max 6)	(max 6)
New matched	0	0	0	0	1
New unmatched	1	0	0	0	0
Programs:					
Number of programs affected	20	15	23	15	19
Applicant-proposing result preferred	8	0	12	1	10
Current NRMP result preferred	12	15	11	14	9
Difference in result by rank number					
5 or fewer ranks	5	3	9	6	3
6–10 ranks	5	3	3	5	3
11–15 ranks	0	5	1	3	1
More than 15 ranks	9	4	6	0	11
	(max 178)	(max 36)	(max 31)		(max 191)
Programs with new position(s) filled	0	0	2	1	1
Programs with new unfilled position(s)	1	0	2	0	0

TABLE 3—DIFFERENCE IN RESULT WHEN ALGORITHM CHANGED FROM PREEXISTING SPECIALTY MATCH TO APPLICANT-PROPOSING

Year	Difference
1991	none
1992	2 applicants improve, 2 programs do worse
1993	2 applicants improve, 2 programs do worse
1994	none
1996	none

Doesn’t really matter!

Prior Theory

Pittel 1989: In 1-to-1 markets with n participants on both sides, and **complete**, **uniformly random** preferences:

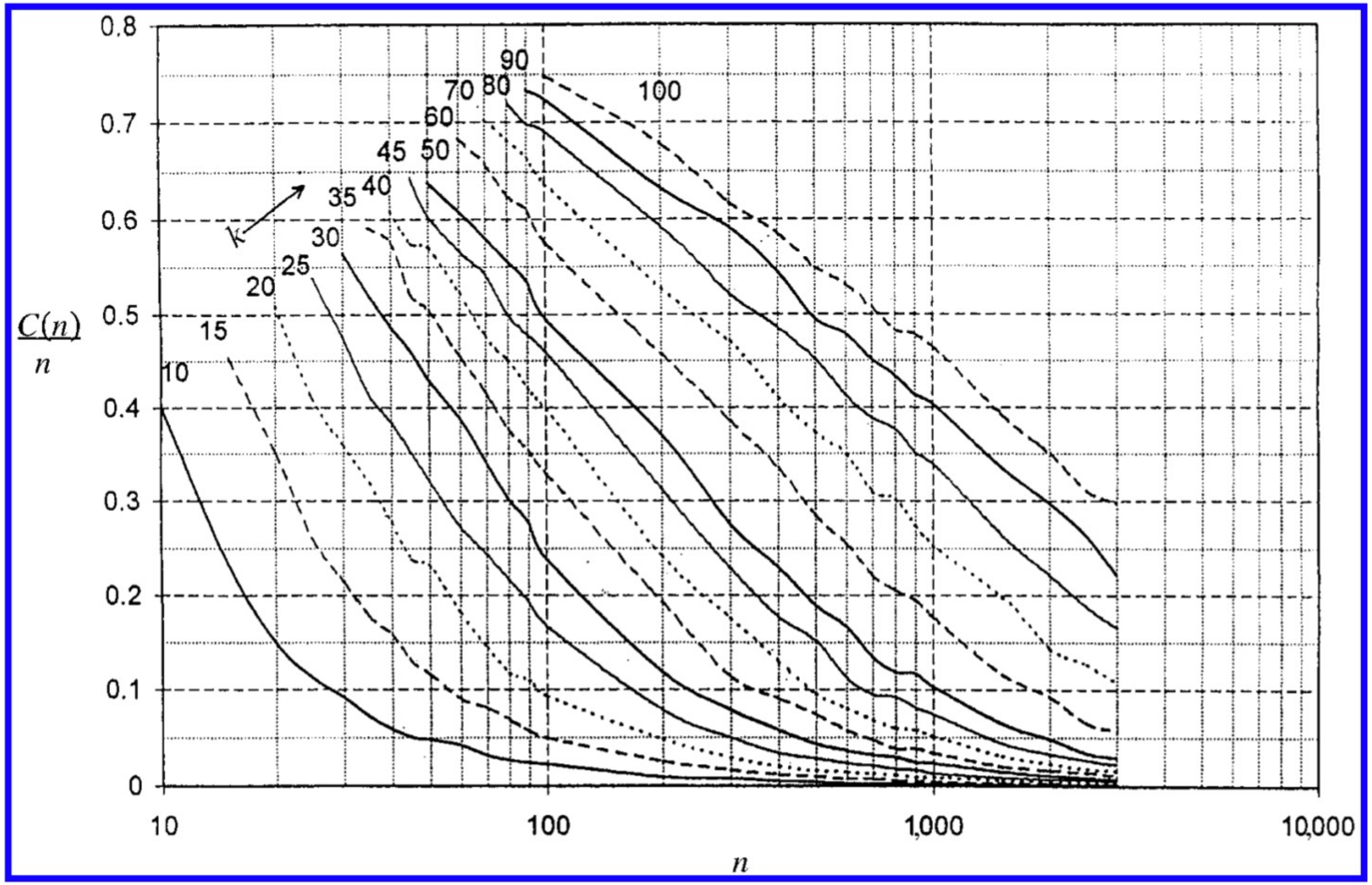
- the average rank of the proposing side is $\approx \log(n)$
- the average rank of the receiving side is $\approx n/\log(n)$

$n = 100$
4.6
21.7

In this case, it really matters which side proposes!

Not true if one side has perfectly correlated preferences.

Simulation Results: Incomplete Lists



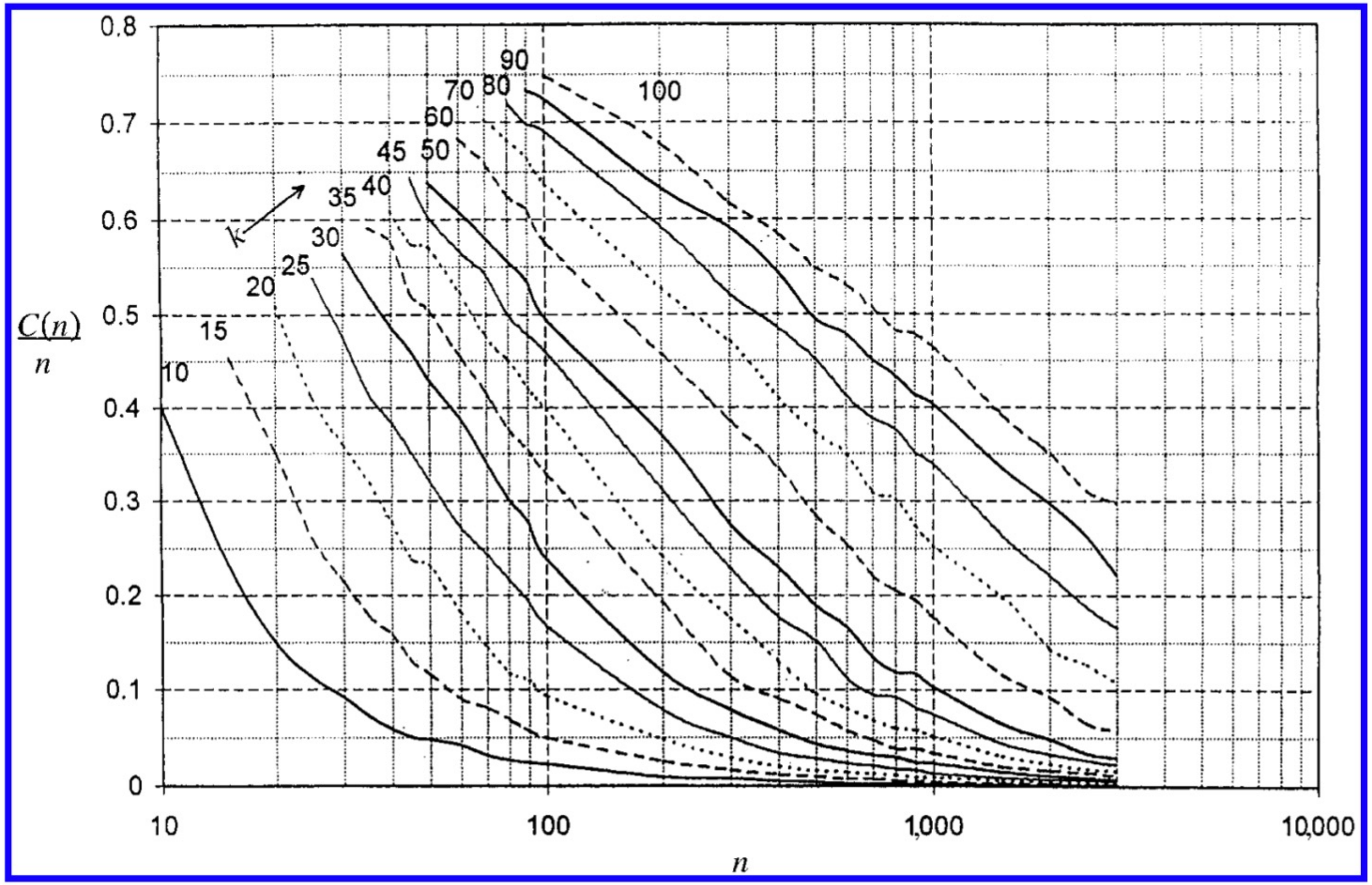
Conjecture:

In 1-to-1 uniform random matching markets where students list only k choices, as $n \to \infty$, the fraction with multiple stable partners $\to 0$.

Proven by Immorlica and Mahdian (2005)

Many-to-one markets: Kojima and Pathak (2009).

Simulation Results: Incomplete Lists



Intuition:

A school has different stable partners if (and only if) it can trigger a “rejection chain” that returns to it.

With short lists, some schools will have unfilled positions. Rejection chains are likely to reach these schools before returning to the original one.

Re-evaluating Balanced Market

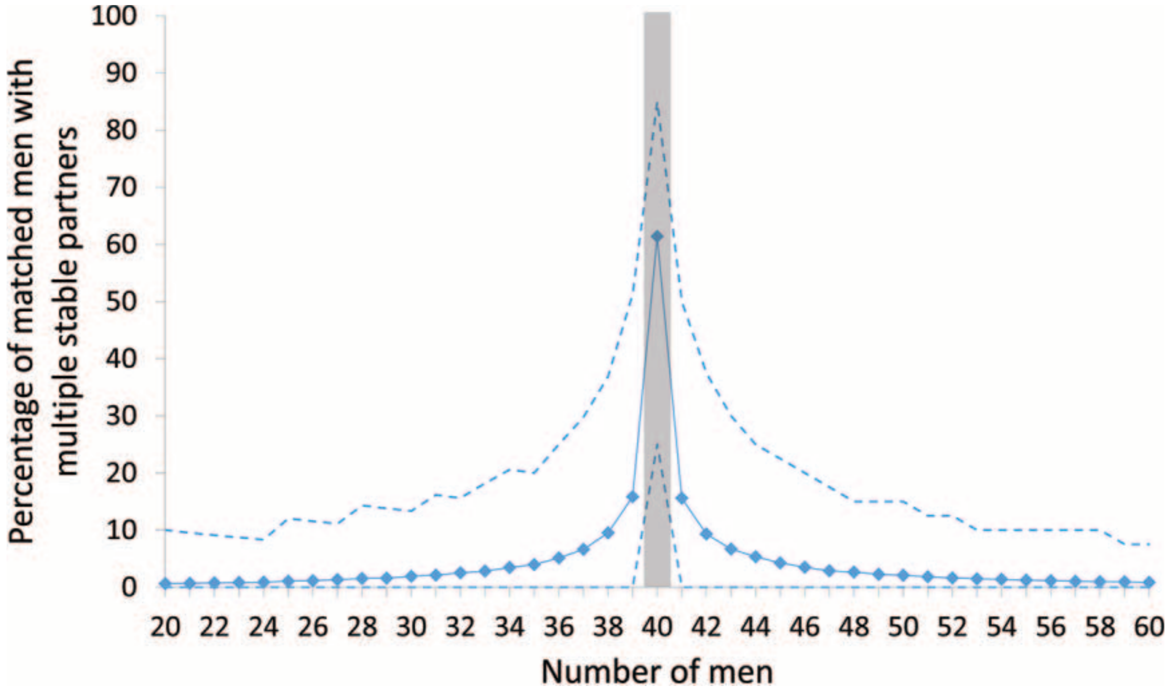


FIG. 1.—Percentage of men with multiple stable partners, in random markets with 40 women and a varying number of men. The main line indicates the average over 10,000 realizations. The dotted lines indicate the top and bottom 2.5th percentiles.

Ashlagi, Kanoria, Leshno (2017)

What about non-uniform random preferences?

Azevedo and Leshno (2016):

When the capacity of each school is large, there is generically a unique stable matching.

One Extra Student Can Make a Big Difference!

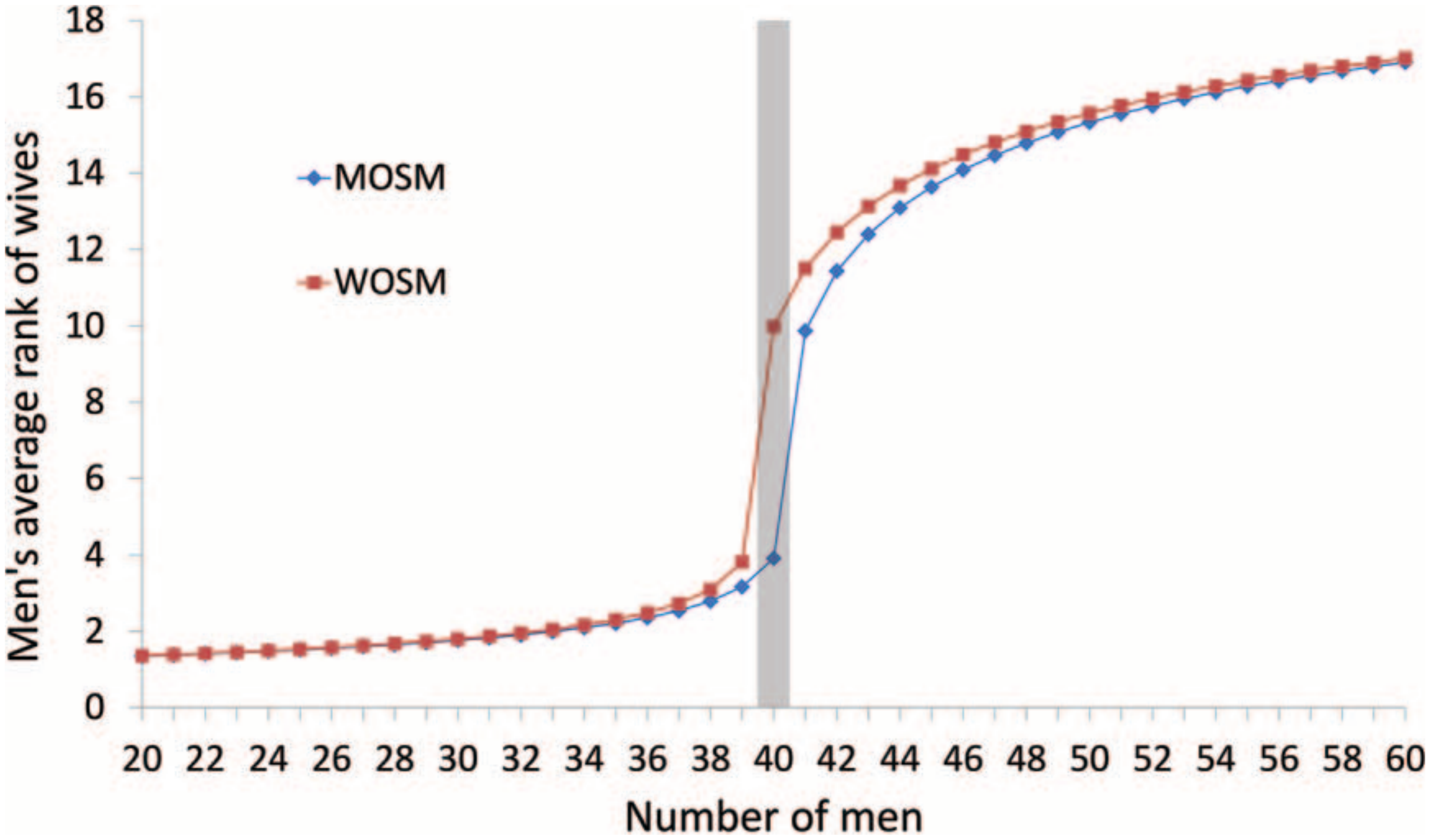


FIG. 2.—Men's average rank of wives under MOSM and WOSM in random markets with 40 women and a varying number of men. The lines indicate the average over 10,000 realizations.

Ashlagi, Kanoria, Leshno (2017)

An Open Problem

Consider two 1-to-1 matching markets with complete random preferences

- Market A: *n* men and *m* women, women propose.
- Market B: *n+1* men and *m* women, men propose.

Conjecture: men's average rank is always lower (better) in Market A.

Potential Conclusion:

competition is more important than choice of algorithm.

Many Student in the Same Priority Class

Two notions of blocking pair:

- **Weak** blocking pair: student prefers school, and has equal priority to another assigned student.
- **Strong** blocking pair: student prefers school, and has higher priority than another assigned student.

Two notions of stability:

- **Strong** stability (no weak blocking pairs) May not exist! NP-Hard to determine.
- **Weak** stability (no strong blocking pairs) Always exist

1	2	3	4	5	6
A	B	A	A	B	A
B	A	B	B	A	B

A	B
1	3,5
2, 3, 4	1,2,4,6
5, 6	

How To Find a Weakly Stable Matching?

1	2	3	4	5	6
A	B	A	A	B	A
B	A	B	B	A	B

A	B
1	3,5
2, 3, 4	1,2,4,6
5, 6	

Tiebreaking Rule

1	2	3	4	5	6
A	B	A	A	B	A
B	A	B	B	A	B

A	B
1	3
2	5
3	1
4	2
5	4
6	6

Then apply standard student-proposing Deferred Acceptance.

Every weakly stable matching is stable in the market resulting from some tiebreaking rule.

In “simple” many-to-one matching markets,

Group Work:
Which properties hold for weakly stable matchings in markets with indifferences?

1. Stable matchings always exist. ✓
2. The set of assigned students and assigned positions is the same for every stable matching. ✗
3. There is a student-optimal stable match. ✗
4. Student-proposing DA finds a stable match. ✓
5. Student-proposing DA is truthful for students. ✓
6. Student-proposing DA is population monotonic. ✓

Can We Optimize Over Set of Stable Matches?

For example, may want to find a (weakly) stable matching that matches the maximum possible number of students.

Concerns:

1. NP-complete (Manlove 1999).
2. Not truthful.
3. Complex to explain, verify.

A	B
1, 2	1, 2

1	2
A	A
B	B

1	2
A	A
B	

How to Break Ties?

Data from NYC

Table 1—Tiebreaking for Grade 8 Applicants in NYC in 2006–2007

Choice	Deferred acceptance single tiebreaking DA-STB (1)	Deferred acceptance multiple tiebreaking DA-MTB (2)	Student-optimal stable matching (3)	Improvement from DA-STB to student-optimal	Number of students (4)
1	32,105.3 (62.2)	29,849.9 (67.7)	32,701.5 (58.4)	+1	633.2 (32.1)
2	14,296.0 (53.2)	14,562.3 (59.0)	14,382.6 (50.9)	+2	338.6 (22.0)
3	9,279.4 (47.4)	9,859.7 (52.5)	9,208.6 (46.0)	+3	198.3 (15.5)
4	6,112.8 (43.5)	6,653.3 (47.5)	5,999.8 (41.4)	+4	125.6 (11.0)
5	3,988.2 (34.4)	4,386.8 (39.4)	3,883.4 (33.8)	+5	79.4 (8.9)
6	2,628.8 (29.6)	2,910.1 (33.5)	2,519.5 (28.4)	+6	51.7 (6.9)
7	1,732.7 (26.0)	1,919.1 (28.0)	1,654.6 (24.1)	+7	26.9 (5.1)
8	1,099.1 (23.3)	1,212.2 (26.8)	1,034.8 (22.1)	+8	17.0 (4.1)
9	761.9 (17.8)	817.1 (21.7)	716.7 (17.4)	+9	10.2 (3.1)
10	526.4 (15.4)	548.4 (19.4)	485.6 (15.1)	+10	4.7 (2.0)
11	348.0 (13.2)	353.2 (12.8)	316.3 (12.3)	+11	2.0 (1.1)
12	236.0 (10.9)	229.3 (10.5)	211.2 (10.4)		
Unassigned	5,613.4 (26.5)	5,426.7 (21.4)	5,613.4 (26.5)	Total:	1,487.5

Notes: Data from the main round of the New York City high school admissions process in 2006–2007 for students requesting an assignment for grade 9 (high school). Column 1 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with single tiebreaking (DA-STB). Column 2 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with school-specific tiebreaking. Column 3 reports the average choice received distribution of applicants in a student-optimal stable matching, which is computed from DA-STB followed by stable improvement cycles. Column 4 reports the average number of students and how many places on their rank order list students improve in the student-optimal stable matching relative to the matching produced by DA-STB. Columns 1, 2, 3, and 4 are based on 250 random draws. Simulation standard errors are reported in parentheses.

Data from Boston

Table 3—Tiebreaking for Elementary School Applicants in Boston in 2006–2007

Choice	Deferred acceptance single tie-breaking DA-STB (1)	Deferred acceptance multiple tie-breaking DA-MTB (2)	Student-optimal stable matching (3)	Improvement from DA-STB to student-optimal	Number of students (4)
1	2,251.8 (8.4)	2,157.3 (13.4)	2,256.6 (8.2)	+1	4.6 (2.6)
2	309.8 (10.3)	355.5 (12.0)	307.4 (10.0)	+2	1.2 (1.1)
3	154.9 (7.9)	189.3 (10.1)	154.0 (7.7)	+3	0.5 (0.7)
4	59.7 (5.5)	76.1 (7.0)	58.7 (5.5)	+4	0.3 (0.5)
5	27.4 (4.5)	34.1 (4.8)	27.0 (4.4)	+5	0.0 (0.1)
6	4.9 (1.9)	6.0 (2.5)	4.9 (1.9)	+6	0.0 (0.1)
7	2.6 (1.4)	2.8 (1.6)	2.5 (1.4)	+7	0.0 (0.1)
8	1.9 (1.2)	0.9 (0.9)	1.9 (1.2)	+8	0.0 (0.1)
9	1.2 (1.1)	0.4 (0.6)	1.2 (1.0)	+9	0.0 (0.0)
10	0.3 (0.6)	0.1 (0.2)	0.3 (0.5)		
Unassigned	112.4 (4.6)	104.6 (4.5)	112.4 (4.6)	Total:	6.5

Notes: Data from Boston Public Schools' student assignment process in round 1 in 2006–2007 for students requesting an assignment for grade K2 (elementary school). Column 1 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with single tiebreaking (DA-STB). Column 2 reports the average choice received distribution of applicants from the student-proposing deferred acceptance algorithm with school-specific tiebreaking. Column 3 reports the average choice received distribution of applicants in a student-optimal stable matching, which is computed from DA-STB followed by stable improvement cycles. Column 4 reports the average number of students and how many places on their rank order list students improve in the student-optimal stable matching relative to the matching produced by DA-STB. Columns 1, 2, 3, and 4 are based on 250 random draws. Simulation standard errors are reported in parentheses.

Group Work:

1. What do you notice?
2. What questions do you have about this data?
3. Based on this data, which tiebreaking procedure would you choose?