Lotteries for Shared Experiences

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We consider a setting where tickets for an experience are allocated by lottery. Each agent belongs to a group, and cares not only about her own allocation, but also about the allocation of other members of her group. In particular, a group is successful if and only if its members receive enough tickets for everyone to participate. We say that a lottery is *efficient*, when it maximizes the number of agents in successful groups, and *fair*, when it gives every group the same chance of success. An ideal lottery would be simultaneously efficient and fair. We study the efficiency and fairness of three mechanisms.

The most widespread mechanism is the *Individual Lottery*. In it, each agent chooses a number of tickets to request. Then agents are ordered uniformly at random and sequentially allocated tickets until none remain. We discuss two deficiencies of this mechanism: (i) large groups have a significant advantage over small groups, and (ii) multiple members from a group may be awarded, resulting in wasted tickets. We show that these issues may lead to arbitrarily unfair and inefficient outcomes.

One alternative is the *Group Lottery*: agents report their groups, then groups are ordered uniformly at random and sequentially allocated tickets until none remain. This mechanism corrects both deficiencies above, and therefore is approximately fair and approximately efficient. However, it requires verifying identities of each group member, which may be too cumbersome for many applications.

Finally, we introduce the *Weighted Individual Lottery*. This is an Individual Lottery biased against agents with large requests. Although it's still possible to have multiple winners in a group, this simple modification drastically reduces the chance of this happening. As a result, this mechanism is approximately fair and approximately efficient, and similar to the Group Lottery when there are many more agents than tickets.

1 INTRODUCTION

1.1 Motivation

Matching models typically assume agents care only about their own allocation. In practice, agents also care about others' allocations: couples entering residency may wish to be matched to programs in the same region, siblings may wish to attend the same school, and friends may want to share a hiking trip. Theory offers little guidance on how to design mechanisms to accommodate such preferences, and in practice a variety of ad-hoc solutions are used.

This paper studies a special case of this problem, in which there is a homogeneous good. In this paper we refer to such goods as tickets. Each agent belongs to a group, and is successful if and only if members of her group receive enough tickets for everyone in the group. One example of such a setting is the allocation of discounted Broadway tickets. While some people may be happy going alone, most prefer to share the experience with others. Recognizing this fact, theaters typically allow each applicant to request multiple tickets. On the morning of the show, applicants are randomly ordered and sequentially given the number of tickets that they requested, until no more remain. We call this the *Individual Lottery*.

1.2 Concerns with the Individual Lottery

To understand why the individual lottery may be unsatisfactory, it is important to note that members of a group may submit separate applications. As we will see, this could lead to unfair outcomes:

- *Multiple entries.* Members of large groups benefit from the fact that each individual can submit an application. Thus, a couple might have approximately twice the chance of success compared to an individual applicant.

In addition, the Individual Lottery suffers from several potential sources of inefficiency. We focus on *over-allocation*, that is, when a group receives more tickets than needed. This might occur for two reasons:

- *Multiple winners.* If both members of a couple apply for and win two tickets, the couple has been awarded four tickets, despite only needing two.
- *Inflated demand.* Because there is no penalty for submitting a large request, an individual may request and receive two tickets.

Both cases result in inefficiency as tickets are allocated but not claimed.

A trivial example where this mechanism performs poorly is as follows. There is one large group and many groups of a single agent. Each agent request the total number of tickets available. Under the Individual Lottery there will be only one group awarded. Hence, if the number of groups is large, then with high probability all tickets are allocated to a single agent. Moreover, the success probability of each group is proportional to its size, giving significant advantage to the large group.

In the example above problems stem from inflated demand. In practice, many applications include a limit ℓ on requests. We show that this modification can lead to further problems. If the designer sets the request limit to be smaller than the size of some groups, then these large groups are disadvantaged, because now they require multiple winners to be successful. Furthermore, if there are many such groups, then most tickets could go to unsuccessful groups, resulting in arbitrary inefficiency. To mitigate this, the designer might consider raising the request limit. Unfortunately, this re-introduces the concern of inflated-demand.

Having identified a set of concerns, it remains to understand their importance. How inefficient and unfair can the Individual Lottery be? If these problems are significant, are there alternative mechanisms that circumvent them?

1.3 Overview of Model and Results

We tackle these questions using a model with k identical tickets. The set of agents is partitioned into a set of groups, and agents have *dichotomous preferences*: an agent is successful if and only if members of her group receive enough tickets for everyone in the group. Because there are only k tickets, the designer can make at most k agents successful. We measure the efficiency of a (randomized) allocation to be the expected number of successful agents, divided by k: if this is at least β, then the allocation is *β-efficient.* A randomized allocation is *fair* if each agent has the same success probability, and *β-fair* if for any pair of agents, the ratio of their success probabilities is at least β.

Given these definitions, we seek (random) allocations that are both approximately efficient and approximately fair. Proposition 4.1 establishes the limits of what can be achieved without any assumptions on group size or number of tickets. It says that there always exists $1/2$-efficient allocation that is fair, but for any $\epsilon > 0$, there are examples where any allocation that is $(1/2 + \epsilon)$-efficient is not even ϵ-fair.

In these bad examples, most groups require a majority of all tickets in order to succeed. In practice, it is often the case that the size of each group is small relative to the number of tickets being allocated. Thus, the majority of our analysis provides performance bounds that depend on the maximum group size and the number of tickets. In addition, we treat the group structure as private information, unknown to the designer. Thus, our goal is to design a mechanism such that when agents act selfishly and the maximum group size is much smaller than the number of tickets, the resulting allocation is both approximately efficient and approximately fair.

Unfortunately, Theorem 4.5 establishes that the Individual Lottery may lead to arbitrarily inefficient and unfair outcomes. In fact, even when all agents report their group size the waste due to over-allocation may be severe. Furthermore, groups will be allocated with probability roughly proportional to their size, so small groups are at a significant disadvantage. These problems exist even if each group needs only a small fraction of the number of tickets.

An alternative approach is to simply ask agents to declare the other members of their group. This allows the designer to avoid over-allocation, the main issue under the Individual Lottery. Moreover, if the designer randomly selects among groups, the resulting allocation should be fair. This is the idea behind the Group Lottery, which we show in Proposition 4.8 incentivizes agents to report truthfully. Because this approach greedily solves a knapsack allocation problem, it is neither perfectly efficient nor perfectly fair, but Theorem 4.9 establishes that it is approximately so if the maximum group size is much less than the number of tickets.

Adopting the Group Lottery would require front-end changes (applicants would have to provide information about their groups) as well as operational changes at event venues (theaters would have to check to see that each ID matched the ticket). Because this may prove unpalatable, we propose an alternative that maintains the same user interface, but changes how requests are processed. The idea is to allow agents to request as many tickets as they wish, but to give those with larger requests a lower chance of being allocated. This eliminates the incentive of inflate demand. If individuals choose to request tickets for their full group, it also eliminates under-allocation. The possibility of multiple winners from the same group remains, but overall efficiency should be higher than under the Individual Lottery.

To make this allocation fair, we choose a particular method for biasing the lottery against large requests: we sequentially select individuals with probability inversely proportional to their request. We call this approach the *Weighted Individual Lottery*. In the Weighted Individual Lottery, a group of four individuals who each request four tickets has the same chance of being drawn next as a group of two individuals who each request two tickets. As a result, outcomes are similar to the Group Lottery, with the exception that it might over-allocate if multiple winners are selected from the same group. Theorem 4.16 establishes that the Weighted Individual Lottery is approximately fair and approximately efficient, and similar to the Group Lottery when demand far exceeds supply.

In summary, in contexts where individuals value tickets only if their group members obtain enough of them, the Individual Lottery is both unfair and inefficient. We propose two alternatives. The Group Lottery is approximately fair and efficient, but may require significant restructuring to implement. The Weighted Individual Lottery is slightly less efficient than the Group Lottery, but requires minimal changes from an Individual Lottery implementation, and leads to similar outcomes to the Group Lottery when demand far exceeds supply.

2 RELATED WORK

In the design of an allocation system there are concerns that naturally arises. For example, the designer should avoid systems that unintendedly give systematically advantage to a particular set of agents. In mechanism design this is usually captured by the concept of fairness. At the same time, the system should allocate tickets in an efficient way with respect to the agents' preferences. To this end, the mechanism should be able to elicit individual preference considering the strategic behavior of the agents.

The difficulty of achieving these familiar requirements in the assignment problem has been established by impossibility theorems in many settings. [1] shows that with four agents or more, there is no ordinal mechanism satisfying simultaneously these requirements. Contrasted with this work, we assume that preferences have a group structure where agents care not only about their own allocation but also about their group's allocation.

Several papers study externalities in two-sided matching (i.e. couples in residency matching). Most of these papers obtain negative results: stable matchings may fail to exist ([9]). Although our settings are quite different, this suggests that general externalities are hard to handle. Therefore, we impose structure with the assumption of dichotomous preferences.

There are many contexts where is natural to assume dichotomous preferences: [2] have studied it in two-sided matching markets. [8] study a kidney exchange system where preferences are determined by patient-donor compatibility. [3] work with them in a problem of collective choice.

The problem of allocate homogeneous goods among agents with multi-unit demand has been considered in several works ([5]; [4]). However, with our preference structure previous solutions such as the proportional rule cannot be directly applied.

We now highlight two works on concentration inequalities that are used in our analysis.

[10] considers the setting of sampling without replacement from a finite population, and presents concentration inequalities for the sample sum. This setting is relevant for our study of the Group Lottery, as a uniform order over a set can be generated by sequentially sampling without replacement from the set. In his work, Serfling introduces the martingale structure inherent in the scheme of sampling without replacement. This martingale is key in the proof of Proposition 4.11, which establishes bounds on the expected hitting time for the sample sum.

[?] state a simple bound on the probability that a Poisson random variable deviates from its expectation at least by a given number. We use this result in our analysis of the Weighted Individual Lottery, where we bound the probability that a group has at least r members awarded using a Poisson random variable.

3 THE MODEL

3.1 Agents, Outcomes, Utilities

A designer must allocate $k \in \mathbb{N}$ indivisible identical tickets to a set of $\mathcal{N} = \{1, \ldots, n\}$ agents. A *feasible allocation* is represented by $x \in \{0, 1, \ldots, k\}^n$ satisfying $\sum_{i \in \mathcal{N}} x_i \leq k$, where x_i indicates the number of tickets that agent i receives. We let $\mathcal{X}$ be the set of all feasible allocations.

A *lottery allocation* is a probability distribution π over $\mathcal{X}$, with π_x denoting the probability of allocation x. Let $\Delta(\mathcal{X})$ be the set of all lottery allocations.

The set $\mathcal{N}$ is partitioned into groups according to $\mathcal{G}$: that is, each $G \in \mathcal{G}$ is a subset of $\mathcal{N}$, $\cup_{G \in \mathcal{G}} G = \mathcal{N}$, and for each $G, G' \in \mathcal{G}$ either $G = G'$ or $G \cap G' = \emptyset$. Given agent $i \in \mathcal{N}$, we let $G_i \in \mathcal{G}$ be the group containing i. Agents are successful if and only if the total number of tickets allocated to the members of their group is at least its cardinality. Formally, each $i \in \mathcal{N}$ is endowed with a utility function $u_i : \mathcal{X} \to \{0, 1\}$ given by

$$u_i(x) = \mathbf{1}\left\{\sum_{j \in G_i} x_j \geq |G_i|\right\}. \tag{1}$$

We say that agent i is *successful* under allocation x if $u_i(x) = 1$.

In a slight abuse of notation, we denote the expected utility of agent $i \in \mathcal{N}$ under the lottery allocation π by

$$u_i(\pi) = \sum_{x \in \mathcal{X}} \pi_x u_i(x).$$

3.2 Performance Criteria

We define the utilization of a feasible allocation as

$$U(x) = \frac{1}{k} \sum_{i \in \mathcal{N}} u_i(x). \tag{2}$$

The *expected utilization* of a lottery allocation π is defined as

$$U(\pi) = \sum_{x \in \mathcal{X}} \pi_x U(x) = \frac{1}{k} \sum_{i \in \mathcal{N}} u_i(\pi). \tag{3}$$

A lottery allocation π is *efficient* if $U(\pi) = 1$. It is *β-efficient* if $U(\pi) \geq \beta$.

A lottery allocation π is *fair* if for every $i, i' \in \mathcal{N}$, $u_i(\pi) = u_{i'}(\pi)$. It is *β-fair* if for every $i, i' \in \mathcal{N}$, $u_i(\pi) \geq \beta u_{i'}(\pi)$.

3.2.1 Alternate Fairness Definition. There are other notions of fairness besides the one considered in this paper. For example, *equal treatment of equals* and *group envy-free*. Below we present the natural analog of these in our setting, and discuss their relation to our fairness definition.

Definition 3.1. We say that random allocation π satisfy equal treatment of equals, if for every pair of agents i, j such that $|G_i| = |G_j|$, we have

$$u_i(\pi) = u_j(\pi). \tag{4}$$

This notion is weaker than our fairness definition, this can be seen as every random allocation that is 1-fair also satisfies this notion. Moreover, it's easy to satisfy equal treatment of equals, in particular, the group request outcome of all our mechanisms satisfy it.

For any $x \in \mathcal{X}$, we let $N_G(x)$ be the number of tickets allocated to members of G. For any $\pi \in \Delta(\mathcal{X})$, we let $N_G(\pi)$ be a random variable representing this number. In an abuse of notation, we let $u_G(N) = \mathbb{P}(N \geq |G|)$.

Definition 3.2. Random allocation π is *group envy-free* if no group envies the allocation of another. That is, $u_G(N_G(\pi)) \geq u_G(N_{G'}(\pi))$ for all $G, G' \in \mathcal{G}$.

This notion is neither stronger nor weaker than our fairness definition. Suppose that there is a group of size 1 and another of size 2. The group of size 1 gets one ticket with probability ϵ, and otherwise gets zero tickets. The group of size 2 gets two tickets with probability ϵ and otherwise gets one ticket. This is fair (according to our definition) but not even approximately group envy-free. Conversely, if both groups get one ticket with probability ϵ and zero tickets otherwise, then the allocation is group envy-free but not fair.

However, the conclusions we draw would also hold for this new fairness notion. The group request outcome of the Individual Lottery may not be even approximately group envy-free. The group request outcome of the Group Lottery is group envy-free. The group request outcome of the Weighted Individual Lottery is approximately envy-free with the same approximation factor as in Theorem 4.16.

3.3 Actions and Equilibria

The designer must choose a *mechanism* consisting of an action set A and an allocation function π. The action set $A = \{A_i\}_{i \in \mathcal{N}}$ establishes for each agent i a discrete set A_i of available actions. The allocation function $\pi : \prod_{i \in \mathcal{N}} A_i \to \Delta(\mathcal{X})$ specifies a lottery allocation for each possible *action profile* $\mathbf{a} \in \prod_{i \in \mathcal{N}} A_i$. We use $\pi(\mathbf{a})$ to denote the lottery allocation associated to the action profile $\mathbf{a}$.

Given any subset of agents $S \subseteq \mathcal{N}$, we use A_S to denote $\prod_{i \in S} A_i$, and A_{-S} to denote $\prod_{i \notin S} A_i$. Similarly, given $S \subseteq \mathcal{N}$ and an action profile $\mathbf{a}$, we use $\mathbf{a}_S$ to denote the actions taken by agents in S, and $\mathbf{a}_{-S}$ to denote the actions taken by agents not in S.

Definition 3.3. The actions $\mathbf{a}_{G_i}$ are *dominant for group $G_i \in \mathcal{G}$* if

$$\mathbf{a}_{G_i} \in \arg\max_{\mathbf{a}'_{G_i} \in A_{G_i}} u_i(\pi(\mathbf{a}'_{G_i}, \mathbf{a}_{-G_i})), \tag{5}$$

holds for any set of actions $\mathbf{a}_{-G_i} \in A_{-G_i}$.

The action profile $\mathbf{a}$ is a *dominant strategy equilibrium* if for each group $G \in \mathcal{G}$ the actions $\mathbf{a}_G$ are dominant.

We assume that each agent has an identifier which is known by the designer, which can be used to ensure that agents participate only once in the mechanism. However, the designer does not know the group structure a priori. Furthermore, members within a group can coordinate their actions. Hence, in Definition 3.3 we need to consider all strategies the group can take instead of focusing on the strategies available for one agent.

4 RESULTS

4.1 An impossibility result

It is trivially possible to achieve fairness, for example by never allocating any tickets. The challenge is to achieve (approximate) fairness while maintaining (approximate) efficiency. The following proposition establishes that it is always possible to achieve $1/2$ efficiency while maintaining fairness. Meanwhile, improving beyond $1/2$-efficiency may require abandoning even approximate fairness.

Proposition 4.1. *For any instance, there exists a random allocation that is fair and* $1/2$ *efficient. For any* $\beta, \epsilon > 0$*, there exist instances such that no random allocation is* β*-fair and* $1/2 + \epsilon$ *efficient.*

Proof. We start with the first claim. Given an instance, partition the groups into disjoint batches, such that no two batches can simultaneously be allocated. That is, each batch consists of a set of groups, every group is in exactly one batch, and any two batches involve more than k individual agents. (Such a batching is trivially found by starting with m batches and greedily merging batches until it is no longer possible to do so.) Consider the random allocation induced by selecting a batch uniformly at random, and giving a ticket to every agent in every group in that batch. It is clear that this allocation is fair. Furthermore, the utilization is the average number of agents in a batch, divided by k. The average number of agents in a batch is at least the average number of agents in the two smallest batches, which is greater than $k/2$ by definition. Therefore, the utilization of this allocation exceeds $1/2$.

For the second claim, consider an example in which one group G' has size $s-1$, the $m-1$ remaining groups have size s, and $k = 2s-1$. For any lottery allocation π, let p_G be the expected utility of an agent in group G. Because only one of the groups $\{2, \dots, m\}$ can be satisfied in any deterministic allocation, we know that

$$\sum_{G \neq G'} p_G \leq 1. \tag{6}$$

If the allocation is β-fair, then it must be the case that

$$\beta p_{G'} \leq \min\{p_G\} \leq 1/(m-1), \tag{7}$$

where the second inequality follows from (6). Furthermore, we have

$$U(\pi) = \frac{(s-1)p_{G'} + s\sum_{G \neq G'} p_G}{k} \leq \frac{(s-1)p_1 + s}{2s-1} \leq \frac{s}{2s-1} + \frac{s-1}{2s-1}\frac{1}{\beta(m-1)} \leq \frac{s}{2s-1} + \frac{1}{2\beta(m-1)}.$$

The first inequality follows from (6), and the second from (7). Taking s and m to be large completes the proof. □

In the example used to prove Proposition 4.1, there are many groups that can only be satisfied if given a majority of the tickets. In many contexts, each group requires only a small number of the k available tickets. Inspired by this thought, we let

$$I(\kappa, \alpha) = \left\{(n, k, \mathcal{G}) : \frac{\max_{G \in \mathcal{G}} |G| - 1}{k} \leq \kappa, \frac{k}{n} \leq \alpha\right\}. \tag{8}$$

The parameter κ captures the significance of the "knapsack" structure, and α captures the "abundance" of the good. When $s \ll k$ (that is, κ is small), we might hope for a solution that is both approximately fair and approximately efficient. Unfortunately, Theorem 4.5 establishes that even in this case, the Individual Lottery is both inefficient and unfair. By contrast, Theorem 4.9 and Theorem 4.16 show that the Group Lottery and the Weighted Individual Lottery are approximately fair and approximately efficient.

4.2 Individual Lottery

In this section, we establish that the Individual Lottery performs poorly in terms of efficiency and fairness. In fact, Theorem 4.5 shows the existence of instances where any dominant strategy equilibrium outcome of the Individual Lottery can be arbitrarily inefficient and unfair.

We begin with a brief description of the Individual Lottery. To start, each agent requests a number of tickets. Then agents are placed in a uniformly random order and processed sequentially. Each agent is given a number of tickets equal to the minimum of their request and the number of remaining tickets.[1]

We now introduce notation that allows us to study this mechanism. We let $\mathcal{O}_\mathcal{N}$ be the set of sequences of elements of $\mathcal{N}$ such that each agent appears exactly once. We refer to an element $\sigma \in \mathcal{O}_\mathcal{N}$ as an *order* over agents, with $\sigma_t \in \mathcal{N}$ and $\sigma_{[t]} = \bigcup_{t' \le t} \sigma_{t'}$ denoting the subset of $\mathcal{N}$ that appears in the first t positions of σ.

Next we provide a formal description of the mechanism. The action set is $A = \{1, \ldots, k\}^n$[2]. Given an action profile $\mathbf{a} \in A$ and an order over agents $\sigma \in \mathcal{O}_\mathcal{N}$, we let $x^{IL}(\mathbf{a}, \sigma) \in \mathcal{X}$ be the feasible allocation generated by the Individual Lottery, formally,

$$x^{IL}_{\sigma_t}(\mathbf{a}, \sigma) = \min\left\{a_{\sigma_t}, \max\left\{k - \sum_{i \in \sigma_{[t-1]}} a_i, 0\right\}\right\}, \tag{9}$$

for $t \in \{1, \ldots, n\}$. For any $x' \in \mathcal{X}$, the allocation function of the Individual Lottery is

$$\pi^{IL}_{x'}(\mathbf{a}) = \sum_{\sigma \in \mathcal{O}_\mathcal{N}} \mathbf{1}\left\{x' = x^{IL}(\mathbf{a}, \sigma)\right\} \frac{1}{n!}.$$

In every mechanism that we will study, there is one strategy that intuitively corresponds to truthful behavior, which we refer to as the *group request strategy*. In the Individual Lottery, this is the strategy in which each agent declares his or her group size.

Definition 4.2. In the Individual Lottery, we say that group G follows the group request strategy if $a_i = |G|$ for all $i \in G$.

4.2.1 Incentives. In the introduction, we discussed two potential sources of inefficiency: under-allocation and over-allocation. The former may be a severe issue in the Individual Lottery when agents request less tickets than their group size, that is, $a_i < |G_i|$. However, in this mechanism agents' request do not affect the order in which they are processed, so each agent should request at least its group size. This is formalized in the proposition below.

Proposition 4.3. *In the Individual Lottery, the set of actions $\mathbf{a}_G$ is dominant for group G if and only if $a_i \ge |G|$ for all $i \in G$.*

[1] One might imagine a different processing rule, in which agents whose request exceeds the number of remaining tickets are skipped. The negatives results in Theorem 4.5 would still apply to this rule.

[2] An alternative mechanism might consider a restricted action set $A = \{1, \ldots, \ell\}^n$ with $\ell \le k$, we refer to such mechanism as the *Individual Lottery with limit ℓ*. At the end of the section, we discuss briefly the impact of introducing a request limit ℓ and give a complete analysis of this mechanism in Appendix A.

4.2.2 Performance. Proposition 4.3 implies that under a dominant strategy equilibrium, only the last agent to be processed might be under-allocated. Therefore, over-allocation will be the main source of inefficiency. Over-allocation might occur for two reasons: (i) existence of inflated demand, i.e., $a_i > |G_i|$ or (ii) multiple winners from the same group. We can eliminate (i) by assuming that every group will follow the group request strategy.[3] This motivates the next proposition.

PROPOSITION 4.4. *Let i be any agent, $\mathbf{a}_{-i}$ be an arbitrary set of actions, and $a_i' > a_i \geq |G_i|$. Then for every agent $j \in N$,*

$$u_j(\pi^{IL}(a_i, \mathbf{a}_{-i})) \geq u_j(\pi^{IL}(a_i', \mathbf{a}_{-i})).$$

Note that by Proposition 4.3 the group request strategy is dominant. Therefore, under a group request equilibrium our only concern will be (ii) multiple winners. Furthermore, it follows from Proposition 4.4 that this is the most efficient dominant strategy equilibrium under the Individual Lottery. Hence, one might expect to obtain similar guarantees as the one provided in Proposition 4.1 – especially if α is small. However, Theorem 4.5 show that even in this case, the individual lottery can be arbitrarily unfair and inefficient.

THEOREM 4.5 ("IL IS BAD"). *For any $\alpha, \kappa, \epsilon \in (0, 1)$, there exists an instance in $I(\kappa, \alpha)$ such that any dominant strategy equilibrium outcome of the Individual Lottery is not $\epsilon-$efficient nor $\epsilon-$fair.*

4.2.3 Proof Sketch of Theorem 4.5. We consider an instance with $n = rs$ agents divided into one large group of size s and $s(r-1)$ small groups of size one. Besides, the number of tickets is $k = \lfloor \alpha r \rfloor s$. It can be shown that for r large enough this instance will be in $I(\kappa, \alpha)$. Let agents i, j be such that $|G_i| = 1$ and $|G_j| = s$.

We will start by proving the efficiency guarantee. By Proposition 4.4 it follows that the group request is the most efficient dominant action profile. Therefore, we assume without loss of generality that this action profile is being selected. We will choose $s, r \in \mathbb{N}$ such that the following two things happen simultaneously:

(1) The size of the large group s is small relative to the number of tickets k.
(2) The fraction of tickets allocated to small groups is insignificant.

This implies our result as the utilization in this system is

$$\frac{1}{k} \sum_{i' \in N} u_{i'}(\pi^{IL}(\mathbf{a})) = \frac{s(r-1)u_i(\pi^{IL}(\mathbf{a}))}{k} + \frac{su_j(\pi^{IL}(\mathbf{a}))}{k} \leq \frac{s(r-1)u_i(\pi^{IL}(\mathbf{a}))}{k} + \frac{s}{k}. \tag{10}$$

The inequality follows as utilities are upper bounded by 1. Because the group request action profile is being selected, the first term above is equal to the fraction of tickets allocated to small groups. Observe that $s/k = 1/\lfloor \alpha r \rfloor$. Hence, to ensure that (1) holds it suffices to have r growing to infinity. Meanwhile, we show the following upper bound on utility of agent i:

$$u_i(\pi^{IL}(\mathbf{a})) \leq \frac{k}{s^2} \leq \frac{\alpha r}{s}. \tag{11}$$

The intuition behind this bound is as follows. If we restrict our attention only to agents in G_i and G_j, then we know that i will get a payoff 0 unless it's processed after at most $k/s - 1$ members of G_j. Because the order over agents is uniformly distributed, this event occurs with probability

$$\frac{k/s}{s+1} \leq \frac{k}{s^2}.$$

[3] Our model assumes that agents are indifferent between all allocations that allow all members of their group to receive a ticket. In practice, there may be reasons to believe that groups will follow the group request strategy (i.e. if each ticket has a cost).

From the first inequality in (11), it follows that

$$\frac{s(r-1)u_i(\pi^{IL}(\mathbf{a}))}{k} \leq \frac{r-1}{s}.$$

Hence, if we let $s = s(r)$ to be such that $r/s \to 0$ as r grows, then (2) holds.

We now turn to the fairness guarantee. To this end, we use a trivial upper bound on the utility of agent j, based on the fact that the first agent to be processed always obtains a payoff of 1. Thus,

$$u_j(\pi^{IL}(\mathbf{a})) \geq \frac{s}{n} = \frac{1}{r}. \tag{12}$$

Note that this lower bound is independent of s, and is attained when all agents in small groups request k tickets. Combining the bound above and the second inequality in (11), we obtain

$$\frac{u_i(\pi^{IL}(\mathbf{a}))}{u_j(\pi^{IL}(\mathbf{a}))} \leq \frac{\alpha r^2}{s}. \tag{13}$$

Therefore, if we choose $s = s(r)$ such that $r^2/s(r) \to 0$ as r grows, then the right side goes to 0 as we take the limit. The full proof of Theorem 4.5 is located in Appendix A.

4.2.4 Extension. The natural extension of this mechanism is the Individual Lottery with limit ℓ. Formally, the only difference is the action set, now $A = \{1, \ldots, \ell\}^n$, where ℓ is chosen by the designer. We show in Proposition A.3 in Appendix A that the worst case performance of the Individual Lottery with limit ℓ is still arbitrary bad. However, in practice there are tradeoffs involved, imposing a limit prevents inflated demand but may harm larger groups. The latter hurts fairness and may also affect efficiency if there are many large groups. Thus, the net effect of imposing a limit "in practice" varies: there are examples where each is better than the other. We give a complete analysis of this mechanism in Appendix A.

Of course, the example presented in Theorem 4.5 is an extreme case that we shouldn't see too often in practice. However, it illustrates the major issues of the Individual Lottery. As we will see in our analysis of the Group Lottery and Weighted Individual Lottery, it is possible to derive performance guarantees even under these extreme cases.

4.3 Group Lottery

The Group Lottery (*GL*) is similar to the Individual Lottery but now groups are processed instead of agents. The action of each agent correspond to a subset of agents, this allow them to identify the members of their group. We say that a group $S \subseteq \mathcal{N}$ is valid if all its members declared the group S. Valid groups are placed in a uniformly random order and processed sequentially (agents that are not part of a valid group will not receive tickets). When a group is processed, if enough tickets remain then every member of the group is given one ticket. Otherwise, members of the group receive no tickets and the lottery ends.

We proceed to formalize this. The set of actions is the power set of $\mathcal{N}$. Given an action profile $\mathbf{a}$, we call a set of agents $S \subseteq \mathcal{N}$ a *valid group* if for every agent $i \in S$ we have that $a_i = S$. In order to define the allocation function, we will first define a function τ that will let us to characterize the number of valid groups that obtain their full request.

Definition 4.6. Fix a set $\mathcal{N}$ and a size function $|\cdot| : \mathcal{N} \to \mathbb{N}$. For any $c \in \mathbb{N}$ and $\sigma \in \mathcal{S}_{\mathcal{N}}$ satisfying $\sum_t |\sigma_t| \geq c$, define

$$\tau(c, \sigma) = \min\left\{T \in \mathbb{N} : \sum_{t=1}^{T} |\sigma_t| \geq c\right\}. \tag{14}$$

Fix an arbitrary action profile $\mathbf{a}$ and the resulting set of valid groups V. For any order $\sigma \in \mathcal{O}_V$, we let $\tau = \tau(k+1, \sigma)$ be as in Definition 4.6 where the size function is the cardinality of the valid group. Intuitively, $\tau - 1$ correspond to the number of valid groups that are processed and obtain their full request. We define

$$x_i^{GL}(\mathbf{a}, \sigma) = \sum_{j=1}^{\tau-1} \mathbf{1}\left\{i \in \sigma_j\right\}. \tag{15}$$

For any $x' \in \mathcal{X}$, the allocation function of the Group Lottery is

$$\pi_{x'}^{GL}(\mathbf{a}) = \sum_{\sigma \in \mathcal{O}_V} \mathbf{1}\left\{x' = x^{GL}(\mathbf{a}, \sigma)\right\} \frac{1}{|V|!}.$$

As in the Individual Lottery, we define the group request strategy which intuitively corresponds to truthful behavior.

Definition 4.7. In the Group Lottery, group $G \in \mathcal{G}$ follows the group request strategy if $a_i = G_i$ for all $i \in G$.

4.3.1 Incentives. In contrast with the Individual Lottery, in the Group Lottery the group request is the only dominant strategy. This motivates the assumption of a group request action profile when analyzing the performance of this mechanism in the next section. We state this result in the next proposition.

Proposition 4.8. *In the Group Lottery, following the group request is the only dominant strategy.*

The intuition behind Proposition 4.8 is as follows. Potential profitable deviations for group G include declaring a superset or split its members into two or more groups. We argue that in both cases the group request is weakly better. First, the competition faced is the same since other valid groups are not affected. Secondly, if there are at least $|G|$ tickets remaining and a valid group containing a member of G is processed, then under the group request G gets a payoff of 1. This might not be true under the alternatives strategies.

4.3.2 Performance. We next argue that the Group Lottery is approximately fair and approximately efficient. Of course, the Group Lottery is not perfectly efficient, as it solves a packing problem greedily. In other words, the random order over groups may determine an allocation that is not maximal in terms of utilization. It is worth noting that under the group request equilibrium in the Group Lottery there is no over-allocation or inflated-demand.

Similarly, it is not perfectly fair, as once there are only a few tickets left, small groups still have a chance of being allocated but large groups do not. Thus, in the Group Lottery smaller groups are always weakly better than larger groups. This is formally stated in Lemma B.1 located in Appendix B.

Theorem 4.9 ("GL is Good"). *Fix $\kappa, \alpha \in (0,1)$. For every instance in $I(\kappa, \alpha)$, the group request equilibrium outcome of the Group Lottery is $(1-\kappa)$-efficient and $(1-2\kappa)$-fair.*

4.3.3 Proof Sketch of Theorem 4.9. The efficiency guarantee is based on the fact that for any order over groups, the number of tickets wasted can be at most $\max_G |G| - 1$. Therefore, the tight lower bound on efficiency of the group lottery is $1 - \frac{\max_G |G| - 1}{k} \geq 1 - \kappa$.

We now turn to the the fairness guarantee. We will show that for any pair of agent i, j,

$$\frac{u_i(\pi^{GL}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))} \geq (1 - 2\kappa). \tag{16}$$

Because all groups are following the group request strategy, the set of valid groups is $\mathcal{G}$. Fix an arbitrary agent i. We construct a uniform random order over $\mathcal{G}$ using Algorithm 2: first generate a uniform random order Σ^{-i} over $\mathcal{G} \setminus G_i$, and then extend it to $\mathcal{G}$ by uniformly inserting G_i in Σ^{-i}. Moreover, if groups in $\mathcal{G} \setminus G_i$ are processed according Σ^{-i}, then $\tau(k - |G_i| + 1, \Sigma^{-i})$ represents the last step in which at least $|G_i|$ tickets remains available. Therefore, if G_i is inserted in the first $\tau(k - |G_i| + 1, \Sigma^{-i})$ positions it will get a payoff of 1. This is formalized in the next lemma.

Lemma 4.10. *For any instance in $I(\kappa, \alpha)$ and any agent i, if we let* a *be the group request strategy under the Group Lottery and Σ^{-i} be a uniform order over $\mathcal{G} \setminus G_i$, then*

$$u_i(\pi^{GL}(\mathbf{a})) = \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{m} \leq \frac{k}{n}(1 + \kappa), \tag{17}$$

where $\tau(k - |G_i| + 1, \Sigma^{-i})$ is as in Definition 4.6 using the cardinality of each group as the size function.

Lemma B.1 in Appendix B states that if two groups are selecting the group request strategy under the Group Lottery, then the utility of the smaller group will be at least the utility of the larger group. Therefore, we assume without loss of generality that $|G_i| \geq |G_j|$. From the Lemma 4.10, it follows that

$$\frac{u_i(\pi^{GL}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))} = \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-j})]} \geq \frac{\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]}{\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})]}. \tag{18}$$

To complete the proof, we express the denominator on the right hand side as the sum of the numerator and the difference

$$\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})] - \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})],$$

which reflects the advantage of the small group G_j. We bound this ratio by taking a lower bound on the numerator $\mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]$ and an upper bound on the difference $\mathbb{E}[\tau(k - |G_j| + 1, \Sigma^{-i})] - \mathbb{E}[\tau(k - |G_i| + 1, \Sigma^{-i})]$. Both bounds follow from the lemma below.

Proposition 4.11. *Given a sequence of numbers $\{a_1, \ldots, a_n\}$ such that $a_t \geq 1$, define $\mu = \sum_i a_i / n$ and $\bar{a} = \max a_i$. Let σ be an order over $\{1, \ldots, n\}$. For $k \in \{1, \ldots, \sum_i a_i\}$, we let $\tau = \tau(k, \sigma)$ be as in Definition 4.6 where the size of i is a_i, that is, $|\sigma_t| = a_{\sigma_t}$. If Σ is a uniform random order of $\{1, \ldots, n\}$, then*

$$1 + \frac{k - \bar{a}}{\mu} \leq \mathbb{E}[\tau(k, \Sigma)] \leq \frac{k + \bar{a} - 1}{\mu}. \tag{19}$$

Furthermore, if $k, k' \in \mathbb{N}$ are such that $k + k' \leq \sum_i a_i$ then

$$\mathbb{E}[\tau(k', \Sigma)] + \mathbb{E}[\tau(k, \Sigma)] \geq \mathbb{E}[\tau(k' + k, \Sigma)]. \tag{20}$$

The intuition for Proposition 4.11 is as follows. We can generate the sequence $\{a_{\Sigma_1}, \ldots, a_{\Sigma_n}\}$ by sequentially sampling numbers without replacement from $\{a_1, \ldots, a_n\}$. Then the hitting time $\tau(k, \Sigma)$ indicates the first time their sum equals target k. It's very well known that a similar version of (19) holds when each a_{Σ_t} is sampled with replacement from $\{a_1, \ldots, a_n\}$. That case can be proved by applying Doob's optional sampling theorem to the classic martingale of a sum of iid random variables. To prove (19), we use a similar approach but considering the natural martingale structure in the scheme of sampling without replacement presented in [10]. This is a fairly general result that applies to any hitting times defined as in 4.6 when sampling without replacement, thus, it is interesting on its own.

In Proposition B.2 located in Appendix B we show that improving beyond $1 - \kappa$-efficiency may require abandoning even approximate fairness. To prove this result, we construct an instance where a particular group must be awarded in order to avoid wasting a fraction κ of the tickets. Therefore, to be more efficient than the Group Lottery is necessary to allocate that group more frequently. In

Proposition B.3 located in Appendix B we construct an instance where the fairness of the Group Lottery is arbitrarily close to the guarantee provided in Theorem 4.9.

4.4 Weighted Individual Lottery

In the Weighted Individual Lottery (IW) agents are placed in a random order, distributed according to the definition below, and are processed in this order. As in the individual lottery, each agent is given a number of tickets equal to the minimum of their action and the number of remaining tickets.

Formally, each agent selects an action in $\{1, \ldots, k\}$. For each $\sigma \in \mathcal{O}_{\mathcal{N}}$, we let random order over agents Σ be such that

$$\mathbb{P}(\Sigma = \sigma|\mathbf{a}) = \prod_{t=1}^{n} \frac{1/a_{\sigma_t}}{\sum\limits_{i \in N \setminus \sigma[t-1]} 1/a_{\sigma_i}}. \tag{21}$$

There are several ways to generate Σ. This order can be thought of as the result of sequentially sampling agents without replacement, with probability inversely proportional to the number of tickets that they request. One property that motivates the study of the Weighted Individual Lottery is that when agents declare their group size, every group that has not been drawn is equally likely to be draw next.

Let $\Sigma \in \mathcal{O}_{\mathcal{N}}$ be distributed according to (21). For any $x' \in \mathcal{X}$, the allocation function of the Weighted Individual Lottery is

$$\pi_{x'}^{SPL}(\mathbf{a}) = \sum_{\sigma \in \mathcal{O}_{\mathcal{N}}} \mathbf{1}\left\{x' = x^{IL}(\mathbf{a}, \sigma)\right\} \mathbb{P}(\Sigma = \sigma),$$

with x^{IL} defined as in (9).

The group request is the same as for the Individual Lottery introduced in Definition 4.7.

4.4.1 Incentives. In this section, we will see that under the Weighted Individual Lottery, there are instances where no strategy is dominant for every group. However, we will argue that if demand significantly exceeds supply, then it's reasonable to assume that groups will select the group request strategy.

We start by showing in Proposition 4.12, that for groups of size three or less the group request is the only dominant strategy. However, as shown in Example 4.13, for larger groups deviating from the group request is potentially profitable.

Proposition 4.12. *In the Weighted Individual Lottery, if $G \in \mathcal{G}$ is such that $|G| \leq 3$, then the group request is the only dominant strategy for G.*

Example 4.13. Consider an instance with n agents and $n - 1$ tickets. We divide the agents into one group of size 4 and $n - 4$ groups of a single agent. If $n \geq 17$, then the optimal strategy for the large group will depend on the action profile selected by the small groups. In particular, if the small groups are following the group request strategy, then members of the larger group benefit from each requesting 2 tickets instead of 4. The analysis of this example is located in Appendix C.

In the example above, when $n \leq 16$, it's actually optimal for the large group to play the group request. Thus, this deviation is only profitable when $n \geq 17$, and the group success probability is bigger than 92%. In general, when agents request fewer tickets than their group size, their chance of being selected increases, but now multiple agents from the group must be drawn in order to achieve success. This should be profitable only if the chance of each agent being drawn is high.

We formalize this intuition in Conjecture 4.14. Roughly speaking, we conjecture that in scenarios where the success probability of a group is below $1 - 1/e \approx 63\%$, the group request strategy maximizes its conditional expected utility. Proposition 4.15 lends additional support to the conjecture. This proposition establishes that our conjecture holds when restricted to a broad set of strategies. In order to present our conjecture, we need first to introduce some definitions.

In what follows, we fix an arbitrary group G. Given any action profile $\mathbf{a}$, we generate an order over agents Σ using the following algorithm:

ALGORITHM 1 (H). *(1) Draw $\{X_i\}_{i\in\mathcal{N}}$ as i.i.d. exponentials, with $\mathbb{P}(X_i > t) = e^{-t}$ for $t \geq 0$.*
(2) Place agents in increasing order of a_iX_i: that is, output Σ such that

$$a_{\Sigma_1}X_{\Sigma_1} < \cdots < a_{\Sigma_n}X_{\Sigma_n}.$$

From Proposition C.1 in Appendix C, it follows that Σ is distributed according to (21) conditional on $\mathbf{a}$. We will refer to a_iX_i as the *score* obtained by agent i. Note that a lower score is better as it increases the chances of getting awarded.

The usual way to study the incentives of group G, is to find a strategy that maximizes its utility given the actions of other agents. Here, we will assume that G has an additional information: the scores of other agents. Thus, we study the problem faced by G of maximizing its success probability given actions and scores of everyone else. This problem seems to be high-dimensional and very complex, however, we will show that all the information relevant for G can be captured by a sufficient statistic T. Define,

$$T = \inf\left\{t \in \mathbb{R} : \sum_{j\notin G} a_j \mathbf{1}\left\{a_jX_j < t\right\} > k - |G|\right\}. \tag{22}$$

We show in Lemma C.2 located in Appendix C, that G gets a utility of 1 if and only if the sum of the requests of its members whose score is lower than T is at least $|G|$. Therefore, we can formulate the problem faced by G as follows:

$$\begin{array}{rl} \max & \mathbb{P}(\sum_{i\in G} a_i \mathbf{1}\{a_iX_i < T\} \geq |G|) \\ \text{subject to} & a_i \in \{1,\ldots,k\} \quad \forall i \in G. \end{array} \tag{23}$$

Notice that under the group request strategy, the objective value in (23) evaluates to

$$1 - e^{-T}. \tag{24}$$

This follows because G will get a payoff of 1 if and only if at least one of its members has a score lower than T, that is, $\min_{i\in G}\{a_iX_i\} < T$. Moreover, using that $\sum_{i\in G} 1/a_i = 1$, and by the well-known properties of the minimum of exponential random variables, we have that $\min_{i\in G}\{a_iX_i\} \sim Exp(1)$.

Definitions out of the way, we can present our conjecture and the proposition supporting it.

CONJECTURE 4.14. *If $T \leq 1$, then no other strategy yields a higher objective value in* (23) *than the group request.*

From equation (24), we see that the utility yield by the group request is an increasing function of T. Therefore, we can think of T as an indicator of how competitive the market is. Thus, the interpretations of Conjecture 4.14 is that if the market is moderately competitive (G has success probability below $1 - 1/e \approx 63\%$), then the group request is optimal. While we haven't prove the conjecture, we do have a proof for the proposition showing that this is true for a broad subset of strategies. For $r \in \{0,\ldots,|G|-1\}$, we define $\mathcal{B}_r \subseteq A_G$ to be the set of strategies for which the sum of any r requests is less than $|G|$, while the sum of any $r+1$ requests is greater than or equal to $|G|$. Let

$$\mathcal{B} = \bigcup_{r=0}^{|G|-1} \mathcal{B}_r. \tag{25}$$

Proposition 4.15. *If* $T \leq 1$*, then no other strategy in* $\mathcal{B}$ *yields a higher objective value in* (23) *than the group request.*

Note that $\mathcal{B}$ is rich enough such that for any group of size greater than 3, it contains a strategy that is better than the group request for T large enough.

Proof sketch of Proposition 4.15. In this proof, we will say that an agent is awarded if and only if it has a score lower than T.

From (24), it suffices to show that under any strategy in $\mathcal{B}_r$, the objective value in (23) is at most $1 - e^{-T}$. We start by studying a relaxation the problem defined in (23). In this relaxation, the number of times an agent i is awarded follows a Poisson distribution with rate T/a_i, and the total number of times G is awarded follows a Poisson distribution with rate $\sum_{i \in G} T/a_i$. Note that if the set of feasible strategies is $\mathcal{B}_r$, then by Lemma C.2 in Appendix C it follows that G needs to be awarded at least $r + 1$ times. Finally, using a Poisson tail bound, we show that this event happens with probability at most $1 - e^{-T}$. This bound can only be applied if the expected number of times G is awarded is at most $r + 1$. This follows because $T \leq 1$ and Lemma C.3 in Appendix C, which establishes that for any $\mathbf{a}'_G \in \mathcal{B}_r$, $\sum_{i \in G} 1/a'_i \leq r + 1$. □

4.4.2 Performance. We now study the performance of the Weighted Individual Lottery, under the assumption that groups are selecting the group request strategy. We think that this assumption is reasonable for two reasons: (i) for groups of size at most three, the group request is the only dominant strategy, and (ii) for larger groups, we conjecture that in scenarios where its success probability is moderate (at most 63%), the group request strategy is optimal. The main result of this section is Theorem 4.16, which establishes that the Weighted Individual Lottery is approximately efficient and fair.

To state these guarantees, we define for any $x > 0$,

$$g(x) = \frac{1 - e^{-x}}{x}. \tag{26}$$

Theorem 4.16. *Fix* $\kappa, \alpha \in (0, 1)$*. For every instance in* $I(\kappa, \alpha)$*, the group request outcome of the Weighted Individual Lottery is* $(1 - \kappa)g(\alpha)$*-efficient and* $(1 - 2\kappa)g(\alpha)$*-fair.*

These guarantees resemble the ones offered for the Group lottery. Recall that Theorem 4.9 establishes that the Group Lottery is $1-\kappa$-efficient and $1-2\kappa$-fair. The Group Lottery is not perfectly efficient, as the last group to be processed might be under-allocated. Similarly, is not perfectly fair, as once there are only a few tickets left, small groups still have a chance of being allocated but large groups do not. These issues persist under the Weighted Individual Lottery. Moreover, now both guarantees are also multiplied by $g(\alpha)$. This term is present in the efficiency guarantee as the Weighted Individual Lottery still suffers from over-allocation. Because $g(\alpha) \geq 1 - \alpha/2$, when α is close to 0 the guarantees for both mechanisms coincide. This has an intuitive explanation: when the group request action profile is selected in the Weighted Individual Lottery, every group that has not been drawn is equally likely to be draw next. Besides, having a supply-demand ratio α close to 0, implies a small chance of having groups with multiple members awarded. Hence, the Weighted Individual Lottery approximates the Group Lottery.

4.4.3 Proof Sketch of Theorem 4.16. In order to prove the efficiency and fairness guarantees, we need first to introduce a new mechanism: the *Group Lottery with Replacement (GR).* The main difference of this new mechanism with the Group Lottery presented in Section 4.3, is that valid groups can be processed more than once.

Formally, the set of actions, the set of valid groups V, the group request strategy and the allocation rule x^{GL} are defined exactly as in the Group Lottery. However, the allocation function

π^{GR} is different, in particular, this mechanism process valid groups according to a sequence of k elements $\Sigma \in \mathcal{S}_V$, where Σ_t is independently and uniformly sampled with replacement from V. Hence, for any $x' \in \mathcal{X}$, the allocation function of the Group Lottery is

$$\pi_{x'}^{GR}(\mathbf{a}) = \sum_{\sigma \in O_V} \mathbf{1}\left\{x' = x^{GL}(\mathbf{a}, \sigma)\right\} \mathbb{P}(\Sigma = \sigma),$$

with x^{GL} defined as in (15). Having defined this new mechanism, we now present a lemma that will be key in proving both guarantees. This lemma establishes a dominance relation between the Weighted Individual Lottery, the Group Lottery and the Group Lottery with Replacement, when the group request action profile is being selected. As we will see, every agent prefers the Group Lottery to the Weighted Individual Lottery, and the Weighted Individual Lottery to the Group Lottery with Replacement.

Lemma 4.17. *For any instance and any agent $i \in \mathcal{N}$, if* $\mathbf{a}$ *denote the corresponding group request action profile for each mechanism below, then*

$$u_i(\pi^{GR}(\mathbf{a})) \leq u_i(\pi^{SPL}(\mathbf{a})) \leq u_i(\pi^{GL}(\mathbf{a})). \tag{27}$$

The key idea to prove Lemma 4.17 is that the order or sequence used in each of these mechanism can be generated based on a random sequence of agents Σ'. Roughly speaking, each order or sequence is generated from Σ' as follows:

- Group Lottery with replacement: replace every agent by its group.
- Weighted Individual Lottery: remove every agent that has already appeared in a previous position.
- Group Lottery: replace every agent by its group, and then remove every group that has already appeared in a previous position.

Note that because in each mechanism the group request strategy is being selected, whenever a group or agent is being processed, it is given a number of tickets equal to the minimum of its group size and the number of remaining tickets.

This implies that, under the Group Lottery with Replacement, a group could be given more tickets than needed because one of its members appeared more than once in the first positions of Σ'. This situation is avoided in the Weighted Individual Lottery, hence, making all agents weakly better. Similarly, under the Weighted Individual Lottery, a group could be given more tickets than needed because its members appeared more than once in the first positions of Σ'. This situation is avoided in the Group Lottery, hence, making all agents weakly better. The full proof is located in Appendix C.2.2.

We now turn to the efficiency guarantee. From Lemma 4.17, it follows that for any instance the utilization under the Weighted Individual Lottery is at least the utilization under the Group Request with Replacement. Therefore, it suffices to show that for any instance in $I(\kappa, \alpha)$, the Group Lottery with Replacement is $(1-\kappa)g(\alpha)$-efficient. To this end, we present in Lemma 4.18 a lower bound on the utility of any agent under the Group Lottery with Replacement.

Lemma 4.18. *For any instance in $I(\kappa, \alpha)$ and any agent i, if we let* $\mathbf{a}$ *be the group request under the Group Lottery with Replacement, then*

$$u_i(\pi^{GR}(\mathbf{a})) \geq \frac{k}{n}(1-\kappa)g(\alpha). \tag{28}$$

The proof of Lemma 4.18 is in Appendix C.2.3. This lemma immediately give us the desired lower bound on the utilization of the Group Lottery with replacement.

We now show the fairness guarantee. From Lemma 4.17, we have that for any instance and any pair of agents i, j,

$$\frac{u_i(\pi^{SPL}(\mathbf{a}))}{u_j(\pi^{SPL}(\mathbf{a}))} \geq \frac{u_i(\pi^{GR}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))}. \tag{29}$$

Hence, it suffice to show that the ratio on the right hand side is at least $(1-2\kappa)g(\alpha)$. In Lemma 4.10 we proved an upper bound on the utility of an agent under the Group Lottery. Meanwhile, in Lemma 4.18 we established a lower bound on the utility of an agent under the Group Lottery with Replacement. Combining equation (29), Lemma 4.10 and Lemma 4.18 yields our fairness factor of $(1-2\kappa)g(\alpha)$.

5 DISCUSSION

The Individual Lottery is used in a variety of applications, including the Broadway musical Hamilton and the Big Sur Marathon. Our work highlights two limitations of this mechanism: (i) large groups have a significant advantage over small groups and (ii) over-allocation due to multiple winners in a group. We show that these issues are severe enough to result in outcomes that are arbitrarily inefficient and arbitrarily unfair. Anecdotally, we do see groups with multiple winners in the results of the Big Sur Marathon. Although the information page states that “a single, designated group leader enters the drawing on behalf of the group", there is no enforcement mechanism. In 2019, the lottery winners included two teams titled “Taylor’s" (with leaders Molly Taylor and Amber Taylor, respectively), as well as a team titled “What the Hill?" and another titled “What the Hill?!"[4].

We propose two alternatives that address these issues. In the Group Lottery there is no over-allocation, and we show in Theorem 4.9 that it’s $1-\kappa$-efficient and $1-2\kappa$-fair. The Weighted Individual Lottery requires minimal changes from an Individual Lottery, and we show in Theorem 4.16 that it’s $(1-\kappa)g(\alpha)$-efficient and $(1-2\kappa)g(\alpha)$-fair. We now apply these results using data from two real-world applications. In 2016 the Hamilton Lottery received approximately $n = 10,000$ applications daily for $k = 21$ tickets, with a max group size of $s = 2$.[5] Hence, in this case $\kappa \leq .05$ and $g(\alpha) \geq .99$. By Theorem 4.9 we get that the Group Lottery outcome is at least 95% efficient and 90% fair. Furthermore, Theorem 4.16 gives the same performance guarantees under Weighted Individual Lottery. The 2020 Big Sur Marathon Groups and Couples round allocated $k = 702$ tickets with a maximum group size of $s = 15$, and a total of $m = 1296$ groups[6]. There is no information about the number of agents n, however, we can use the conservative lower bound $n \geq m$. This yields $\kappa \leq .02$ and $g(\alpha) \geq .77$. Theorem 4.9 implies 98% efficiency and 96% fairness for the Group Lottery. Besides, Theorem 4.16 implies 76% efficiency and 74% fairness for the Weighted Individual Lottery.

All of our analysis is conducted under the strong assumption of dichotomous preferences. In practice, the world is more complicated: groups may benefit from extra tickets that can be sold or given to friends, and groups that don’t receive enough tickets for everyone may choose to split up and have a subset attend the event. Despite these considerations, we believe that dichotomous preferences capture the first-order considerations in several markets while maintaining tractability. Moreover, the mechanisms we have proposed, while imperfect, would be practical and offer improvements over the status quo.

Another assumption in our model is that each agent submits at most one application. This raises a practical issue: how do organizations prevent multiple applications in practice? In the

[4]More information about the drawing is available at https://web.archive.org/web/20200407192601/https://www.bigsurmarathon.org/drawing-info/. The list of groups awarded in 2019 is available at http://www.bigsurmarathon.org/wp-content/uploads/2018/07/Group-Winners-for-Website.pdf

[5]Source: https://www.bustle.com/articles/165707-the-odds-of-winning-the-hamilton-lottery-are-too-depressing-for-words.

[6]Source: https://www.bigsurmarathon.org/random-drawing-results-for-the-2020-big-sur-marathon/.

examples mentioned above, to enter the lottery an email or phone number is required. If there is no mechanism to prevent multiple applications, then most solutions will perform badly. In the extreme case, an agent could submit thousands of applications, and obtain a significant fraction of the tickets available. If multiple requests is an issue, we recommend to address this before improving the allocation mechanism.

One exciting direction for future work is to adapt these mechanisms to settings with heterogeneous goods. For example, the lottery for permits to climb Half Dome works as follows. There are 225 permits available each day. Before the hiking season, each applicant enters a number of permits requested (up to a maximum of six), as well as a ranked list of dates that would be feasible. Applicants are then placed in a uniformly random order, and sequentially allocated their most preferred feasible date. This is the natural extension of the Individual Lottery to a setting with with heterogeneous goods, and has many of the same limitations discussed in this paper. It would be interesting to study the performance of (generalizations of) the Group Lottery and Weighted Individual Lottery in this setting.

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A INDIVIDUAL LOTTERY

ALGORITHM 2. *Choose* $S \subset \mathcal{N}$. *Generate*

i. *a uniform random order* $\Sigma^S \in \mathcal{O}_S$,
ii. *a uniform random order* $\Sigma^- \in \mathcal{O}_{\mathcal{N} \setminus S}$.
iii. *a uniform random subset* $P \subset \{1, \ldots, |\mathcal{N}|\}$ *with* $|P| = |S|$.

Generate Σ *from* Σ^S, Σ^-, P *placing elements of* S *in positions* P, *maintaining the order of elements of* S *as given by* Σ^S *and the order of elements of* $\mathcal{N} \setminus S$ *as given by* Σ^-.

PROPOSITION A.1. *Algorithm 2 generates a uniform random order* $\Sigma \in \mathcal{O}_\mathcal{N}$ *: for each order* $\sigma \in \mathcal{O}_\mathcal{N}$, $P(\Sigma = \sigma) = 1/|\mathcal{N}|!$.

PROOF OF PROPOSITION A.1. Fix an order $\sigma \in \mathcal{O}_\mathcal{N}$. Let σ_S and σ_{-S} be the restriction of σ to S and $\mathcal{N} \setminus S$, respectively. For any $i \in \mathcal{N}$, we let p_i be the position of i in σ, that is, $\sigma_{p_i} = i$. We define $P_S = \cup_{i \in S} \{p_i\}$. In order to end with the order σ, it must be that:

- The order Σ generated in step i. is equal to σ_S, which occurs with probability $\mathbb{P}(\Sigma = \sigma_S) = 1/|S|!$.
- The order Σ^- generated in step ii. is equal to σ_{-S}, which occurs with probability $\mathbb{P}(\Sigma^- = \sigma_{-S}) = 1/(|\mathcal{N}| - |S|)!$.
- The random subset P generated in step iii. is equal to P_S, which occurs with probability $\mathbb{P}(P = P_S) = \prod_{j=0}^{|S|-1} (|S| - j)/(|\mathcal{N}| - j)$.

Hence, the probability that the algorithm generates the order σ is

$$\frac{1}{|S|!} \frac{1}{\mathcal{N} - |S|!} \prod_{j=0}^{|S|-1} \frac{|S| - j}{|\mathcal{N}| - j} = \frac{1}{|\mathcal{N}|!}.$$

□

A.1 Incentives

PROOF OF PROPOSITION 4.3. This is a direct consequence of Proposition A.2. Notice that for any given instance with k tickets, the Individual Lottery is equivalent to the Individual Lottery with limit $\ell = k$. Therefore, $r = \lceil |G|/k \rceil = 1$ and our result follows. □

PROOF OF PROPOSITION 4.4. Consider any agent i. We let $\mathbf{a}_{-i} \in A_{-i}$ be an arbitrary set of actions and $a'_i > a_i \geq |G_i|$. We begin by showing that for any order over agents σ, the conditional expected utility of i is the same under both strategies. We consider two possible cases. First, the number of tickets remaining before agent i is processed is a_i or less. Then, under both strategies the allocation of every agent is the same and the payoff of G_i coincide. Second, the number of tickets remaining before agent i is processed is greater than a_i. Then, under both strategies agent i obtains at least $|G_i|$ tickets and the group gets a payoff of 1.

Now, we will show that the utility of every group $G \neq G_i$ is weakly better under a_i. It suffices to show that for any order $\sigma \in \mathcal{O}_\mathcal{N}$,

$$x_j^{IL}((a_i, \mathbf{a}_{-i}), \sigma) \geq x_j^{IL}((a'_i, \mathbf{a}_{-i}), \sigma) \text{ for every agent } j \neq i. \tag{30}$$

Because this holds for any order σ, and the random order over agents used in the Individual Lottery is uniformly distributed, this implies our result. Let $T = T(\sigma)$ be the position of agent i in σ, that is, $T = \{t \in \{1, \ldots, n\} : \sigma_t = i\}$. The allocation of agents $\sigma_1, \ldots, \sigma_{T-1}$ is not affected by the action of i, then (30) holds. A smaller request can only lead to a smaller allocation, hence the allocation of agent i is weakly smaller under a_i. Therefore, the allocation of agents $\sigma_{T+1}, \ldots, \sigma_n$ is weakly greater under a_i as the only difference is due to agent i. □

A.2 Performance

Proof of Theorem 4.5. Consider an instance with $n = rs$ agents divided into one large group of size s and $s(r-1)$ small groups of size one. Besides, the number of tickets is $k = \lfloor \alpha r \rfloor s$. Observe that for any $s, r \in \mathbb{N}$,

$$\frac{k}{n} = \frac{\lfloor \alpha r \rfloor s}{rs} \leq \alpha.$$

Thus, if $(s-1)/k \leq \kappa$ then this instance will be in $I(\kappa, \alpha)$. We claim that a sufficient condition for this is

$$r \geq \frac{\kappa + 1}{\alpha \kappa}. \tag{31}$$

This can be seen as

$$\frac{s-1}{k} = \frac{s-1}{\lfloor \alpha r \rfloor s} \leq \frac{1}{\alpha r - 1} \leq \kappa.$$

In the first inequality we use that for any x, $\lfloor x \rfloor \geq x - 1$ and $(s-1)/s \leq 1$. The last inequality follows from (31).

Let agents i, j be such that $|G_i| = 1$ and $|G_j| = s$. We claim that for any dominant strategy equilibrium $\mathbf{a}$,

$$u_i(\pi^{IL}(\mathbf{a})) \leq \frac{k}{s^2} \leq \frac{\alpha r}{s}. \tag{32}$$

This bound is key to prove both guarantees. We start by proving the efficiency result. The expected utilization in this system is:

$$\begin{aligned}\frac{1}{k} \sum_{i' \in \mathcal{N}} u_{i'}(\pi^{IL}(\mathbf{a})) &= \frac{s}{k}\Big((r-1)u_i(\pi^{IL}(\mathbf{a})) + u_j(\pi^{IL}(\mathbf{a})\Big) \\ &\leq \frac{r-1}{s} + \frac{s}{k} \\ &= \frac{r-1}{s} + \frac{1}{\lfloor \alpha r \rfloor}.\end{aligned}$$

In the inequality we use that $u_j(\pi^{IL}(\mathbf{a})) \leq 1$ and the first inequality in (32). Hence, if we choose $s = s(r)$ such that $r/s(r) \to 0$ as r grows, then the right side goes to 0 as we take the limit.

We now turn to the fairness guarantee. Because the first agent to be processed always get a payoff of 1, we get that

$$u_j(\pi^{IL}(\mathbf{a})) \geq \frac{s}{n} = \frac{1}{r}.$$

Note that this lower bound is independent of s, and is tight when all agents in small groups request k tickets.

Using this and the second inequality in (32), we obtain

$$\frac{u_i(\pi^{IL}(\mathbf{a}))}{u_j(\pi^{IL}(\mathbf{a}))} \leq \frac{\alpha r^2}{s}.$$

Therefore, if we choose $s = s(r)$ such that $r^2/s(r) \to 0$ as r grows, then again the right side goes to 0 as we take the limit.

All that remains is to prove (32). We let Σ be a random order over agents. To generate Σ we use Algorithm 2 from Proposition A.1: set $S = G_i \cup G_j$, and independently generate (i) a uniform random order Σ^S over S, (ii) a uniform random order Σ^- over $\mathcal{N} \setminus S$, and (iii) uniform random positions $P \subseteq \{1, \ldots, |\mathcal{N}|\}$ where agents in S will be placed. By Proposition A.1, the resulting order

Σ is uniformly distributed. Note that i will get a payoff 0 unless it appears in the first k/s positions of Σ^S. Because Σ^S is uniformly distributed this event occurs with probability

$$\frac{k/s}{s+1} \le \frac{k}{s^2}.$$

This implies (32) and concludes our proof. □

A.3 Extension

Proposition A.2. *In the Individual Lottery with limit ℓ, the set of actions $\mathbf{a}_G$ is dominant for group G if and only if $\sum_{i \in S} a_i \ge |G|$ for all $S \subseteq G$ such that $|S| = \lceil |G|/\ell \rceil$.*

Proof of Proposition A.2. Fix an arbitrary agent i. Let $\mathbf{a}_{-G_i} \in A_{-G_i}$ be an arbitrary action profile for agents not in G_i. We let $r = \lceil |G_i|/\ell \rceil$ be the minimum number of members of G_i that must be awarded in the Individual Lottery with limit ℓ in order for G_i to get a payoff of 1. Let $\mathbf{a}_{G_i} \in A_{G_i}$ be any action profile such that $\sum_{j \in S} a_j \ge |G_i|$ for all $S \subseteq G_i$ such that $|S| = r$.

First, we show that for any order $\sigma \in \mathcal{O}_{\mathcal{N}}$ the utility of agent i is maximized under $\mathbf{a}_{G_i}$, that is,

$$u_i(x^{IL}((\mathbf{a}_{G_i}, \mathbf{a}_{-G_i}), \sigma)) \ge u_i(x^{IL}((\mathbf{a}'_{G_i}, \mathbf{a}_{-G_i}), \sigma)), \text{ for every } \mathbf{a}'_{G_i} \in A_{G_i}. \tag{33}$$

Because this holds for any order σ, the expected utility of group G_i will also be maximized by $\mathbf{a}_{G_i}$. Let $T = T(\sigma)$ be the position of the r^{th} member of G_i under σ:

$$T(\sigma) = \min\{t \in \{1, \dots, n\} : \left|\sigma_{[t]} \cap G_i\right| = r\}.$$

If $\sum_{j \in \sigma_{[T]} \setminus G_i} a_j > k - |G_i|$, then for any action profile selected by group G_i its payoff is 0.

If $\sum_{j \in \sigma_{[T]} \setminus G_i} a_j \le k - |G_i|$, then under action profile $\mathbf{a}_{G_i}$ group G_i receives a payoff of 1.

Hence, $\mathbf{a}_{G_i}$ maximizes the payoff of agent i for each σ.

Second, let $\hat{\mathbf{a}}_{G_i}$ be such that $\sum_{j \in S} \hat{a}_j < |G_i|$ for some $S \subseteq G_i$ with $|S| = r$. We will show that there exists an order $\hat{\sigma} \in \mathcal{O}_{\mathcal{N}}$ such that

$$1 = u_i(x^{IL}((\mathbf{a}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma})) > u_i(x^{IL}((\hat{\mathbf{a}}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma})) = 0.$$

This combined with (33) implies that $\hat{\mathbf{a}}_{G_i}$ is dominated by $\mathbf{a}_{G_i}$. We construct $\hat{\sigma}$ in the following way:

- Agents in S are arbitrary placed in the first r positions of $\hat{\sigma}$.
- Agents in $\mathcal{N} \setminus G_i$ are arbitrary placed in positions $r+1, \dots, r+n-|G_i|$ of $\hat{\sigma}$.
- Agents in $G_i \setminus S$ are arbitrary placed in the last $|G_i| - r$ positions of $\hat{\sigma}$.

We begin by proving that if $\mathbf{a}_{G_i}$ is selected then G_i received at least $|G_i|$ tickets, implying that $u_i(x^{IL}((\mathbf{a}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma})) = 1$. To see this note that the number of tickets received by G_i is

$$\sum_{j \in G} x_j^{IL}((\mathbf{a}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma}) \ge \sum_{j \in S} x_j^{IL}((\mathbf{a}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma}) \ge \min\{k, \sum_{j \in S} a_j\} \ge |G_i|.$$

The last inequality follows as $k \ge |G_i|$ and $\sum_{j \in S} a_j \ge |G_i|$. On the other hand, we show that when $\hat{\mathbf{a}}_{G_i}$ is selected then G_i received strictly less than $|G_i|$ tickets and $u_i(x^{IL}((\hat{\mathbf{a}}_{G_i}, \mathbf{a}_{-G_i}), \hat{\sigma})) = 0$. For the sake of contradiction, suppose that $|G_i|$ received at least $|G_i|$ tickets, then

$$\sum_{j \in \mathcal{N}} x_j^{IL}((\mathbf{a}_{G_i}, \hat{\mathbf{a}}_{-G_i}), \hat{\sigma}) = \sum_{j \in G_i} x_j^{IL}((\mathbf{a}_{G_i}, \hat{\mathbf{a}}_{-G_i}), \hat{\sigma}) + \sum_{j \in \mathcal{N} \setminus G_i} x_j^{IL}((\mathbf{a}_{G_i}, \hat{\mathbf{a}}_{-G_i}), \hat{\sigma}) \ge |G_i| + \sum_{j \in \mathcal{N} \setminus G_i} 1 = n.$$

A contradiction, as $k < n$. Note that G_i will get $|G_i|$ or more tickets only if agent in position $r + n - |G_i| + 1$ is awarded, this implies that all agents in the first $r + n - |G_i|$ must also be awarded. □

Proposition A.3. *For any $\alpha, \kappa, \epsilon \in (0,1)$ and $\ell \in \mathbb{N}$, there exists an instance in $I(\kappa, \alpha)$ such that, regardless the action profile selected, the outcome of the Individual Lottery with limit ℓ is not $\epsilon-$efficient nor $\epsilon-$fair.*

Proof of Proposition A.3. Consider a sequence of instances with $n \to \infty$ and a constant k number of tickets. In each instance, there is one group of size 1 and the remaining groups have size $\ell + 1$. We let i be such that $|G_i| = 1$ and j be such that $|G_j| = \ell + 1$. We let Σ be a uniform order over agents.

First, note that regardless of the action profile $\mathbf{a}$, i gets utility 1 if among the first $\lceil k/\ell \rceil$ agents in Σ, and gets utility 0 if after the first k agents. Because Σ is drawn uniformly at random from $O_{\mathcal{N}}$, we have

$$\frac{\lceil k/\ell \rceil}{n} \leq u_i(\pi^{IL}(\mathbf{a})) \leq \frac{k}{n}. \tag{34}$$

Second, because $\ell < |G_j|$, at least two agents from group G_j must be awarded in order for the group to get utility 1. Furthermore, any agent not among the first k will certainly not receive any tickets. Therefore, group G_j gets utility 1 only if some pair of agents from G_j are both among the first k agents. For any pair of agents, the chance that both are among the first k agents is $\binom{n-2}{k-2}/\binom{n}{k}$. Applying a union bound, we see that

$$u_j(\pi^{IL}(\mathbf{a})) \leq \frac{\binom{n-2}{k-2}\binom{\ell+1}{2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}\binom{\ell+1}{2}. \tag{35}$$

Combining the upper bounds derived in (34) and (35), we bound the overall efficiency as follow

$$\frac{1}{k}\sum_{i' \in \mathcal{N}} u_{i'}(\pi^{IL}(\mathbf{a})) \leq \frac{1}{k}\left(\frac{k}{n} + (n-1)\frac{k(k-1)}{n(n-1)}\binom{\ell+1}{2}\right) = \frac{2 + (k-1)(\ell+1)\ell}{2n},$$

which approaches zero as n grows.

Furthermore, (34) and (35) imply that

$$\frac{u_j(\pi^{IL}(\mathbf{a}))}{u_i(\pi^{IL}(\mathbf{a}))} \leq \frac{\frac{k(k-1)}{n(n-1)}\binom{\ell+1}{2}}{\frac{\lceil k/\ell \rceil}{n}} = \frac{k(k-1)}{\lceil k/\ell \rceil (n-1)}\binom{\ell+1}{2},$$

which also approaches zero as n grows. ☐

Proposition A.4. *For any $\ell \in \mathbb{N}$ and any instance such* $\max_{G \in \mathcal{G}} |G| \leq \ell$, *every dominant strategy equilibrium outcome of the Individual Lottery with limit ℓ is $1/\ell-$efficient.*

Proof of Proposition A.4. We let Σ be a uniform order over agents. We claim that if $\mathbf{a}_{G_i}$ is dominant for G_i, then

$$\mathbb{E}[u_i(x^{IL}(\mathbf{a}, \Sigma))] \geq \frac{k}{\ell n}. \tag{36}$$

From this, it follows that if $\mathbf{a}$ is such that all agents follow a dominant strategy, then

$$\mathbb{E}[U(x^{IL}(\mathbf{a}, \Sigma))] = \frac{1}{k}\sum_{i \in \mathcal{N}} \mathbb{E}[u_i(x^{IL}(\mathbf{a}, \Sigma))] \geq \frac{1}{\ell}.$$

We now prove (36). If $|G_i| = 1$, then no matter the reports of others, i succeeds if in the first $\lceil k/\ell \rceil$ positions, which occurs with probability $\frac{\lceil k/\ell \rceil}{n} \geq \frac{k}{\ell n}$.

Otherwise, because $\max_{G \in \mathcal{G}} |G| \leq \ell$ and agents in G_i follow a dominant strategy, i succeeds if any agent from G_i is in the first $\lfloor k/\ell \rfloor$ positions.

If $k/\ell < 2$, then this occurs with probability $\frac{|G_i|}{n} \geq \frac{k}{\ell n}$. Thus, we turn to the case with $\min(|G_i|, k/\ell) \geq 2$. Fix two agents in G_i. The chance that at least one of them is in the first $\lfloor k/\ell \rfloor$ positions is

$$\begin{aligned}\frac{2\lfloor k/\ell \rfloor}{n} - \frac{\binom{n-2}{\lfloor k/\ell \rfloor - 2}}{\binom{n}{\lfloor k/\ell \rfloor}} &\geq \frac{2\lfloor k/\ell \rfloor}{n} - \left(\frac{\lfloor k/\ell \rfloor}{n}\right)^2 \\ &\geq \frac{k/\ell - 1}{n} + \frac{\lfloor k/\ell \rfloor}{n} - \left(\frac{\lfloor k/\ell \rfloor}{n}\right)^2 \\ &= \frac{k}{\ell n} - \frac{1}{n} + \frac{\lfloor k/\ell \rfloor}{n}\left(1 - \frac{\lfloor k/\ell \rfloor}{n}\right).\end{aligned}$$

All that remains is to establish that

$$\frac{\lfloor k/\ell \rfloor}{n}\left(1 - \frac{\lfloor k/\ell \rfloor}{n}\right) \geq \frac{1}{n}.$$

This holds because $\lfloor k/\ell \rfloor \geq 2$ by assumption, and $1 - \frac{\lfloor k/\ell \rfloor}{n} \geq 1 - 1/\ell \geq 1/2$. □

Proposition A.5. *For any $\ell \in \mathbb{N}$ and any instance such that $\max_{G \in \mathcal{G}} |G| \leq \ell$, every dominant strategy equilibrium outcome of the Individual Lottery with limit ℓ is $1/\ell$–fair.*

Proof of Proposition A.5. We construct a random order over agents Σ using Algorithm 2: set $S = G_i \cup G_j$, and independently generate (i) a uniform random order Σ^S over S, (ii) a uniform random order Σ^- over $\mathcal{N} \setminus S$, and (iii) uniform random positions $P \subseteq \{1, \ldots, |\mathcal{N}|\}$ where agents in S will be placed. By Proposition A.1, the resulting order Σ is uniformly distributed.

Without loss of generality, we assume

$$\ell \geq |G_i| \geq |G_j|. \tag{37}$$

We let

$$\tau_i(\Sigma^-) = \tau(k - |G_i| + 1, \Sigma^-), \tag{38}$$

$$\tau_j(\Sigma^-) = \tau(k - |G_j| + 1, \Sigma^-) - \tau(k - |G_i| + 1, \Sigma^-), \tag{39}$$

be as in Definition 4.6 where the size of each agent is its request, that is, $|\sigma_t| = a_{\sigma_t}$. Note that by definition,

$$1 \leq \tau_i(\Sigma^-), \qquad \text{and} \qquad \tau_j(\Sigma^-) \leq |G_i| - |G_j|. \tag{40}$$

In addition, for $s \in \{1, \ldots, |S|\}$ let $T_s(P)$ be the s^{th} smallest value in P, so $T_1(P)$ denotes the first position of Σ containing a member of $G_i \cup G_j$. Note that

$$\mathbb{P}(T_1(P) = t) = \left(\frac{|G_i| + |G_j|}{n}\right) \frac{\binom{n-t}{|G_i|+|G_j|-1}}{\binom{n-1}{|G_i|+|G_j|-1}},$$

which is decreasing in t. From this, it follows that for any Σ^-,

$$\frac{\mathbb{P}(T_1 \leq \tau_i + \tau_j | \Sigma^-)}{\mathbb{P}(T_1 \leq \tau_i | \Sigma^-)} \leq \frac{\tau_i + \tau_j}{\tau_i} \leq 1 + |G_i| - |G_j|, \tag{41}$$

where the second inequality comes from (40). Our final definition is to let

$$A_i = \{\Sigma_1^S \in G_i\}, \qquad A_j = \{\Sigma_1^S \in G_j\}, \tag{42}$$

and note that

$$\mathbb{P}(A_i) = \frac{|G_i|}{|G_i| + |G_j|} = 1 - \mathbb{P}(A_j). \tag{43}$$

Definitions out of the way, we proceed with the proof. Note that

$$\frac{u_j(\pi^{IL}(\mathbf{a}))}{u_i(\pi^{IL}(\mathbf{a}))} = \frac{\mathbb{E}[\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-]]}{\mathbb{E}[\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-]]} \leq \max_{\sigma^-} \frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^- = \sigma^-]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^- = \sigma^-]}.$$

Therefore, to establish $1/\ell$ fairness, it suffices to show that for every Σ^-,

$$\frac{1}{\ell} \leq \frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-]} \leq \ell. \tag{44}$$

We claim that

$$\mathbb{P}(A_i)\mathbb{P}(T_1 \leq \tau_i|\Sigma^-) \leq \mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-] \leq \mathbb{P}(T_1 \leq \tau_i|\Sigma^-). \tag{45}$$

The left inequality follows because whenever $T_1 \leq \tau_i$ and a member of G_i comes before all members of G_j, group G_i gets a payoff of 1. The right inequality follows because the definition of τ_i ensures that G_i can get a payoff of one only if a member of G_i is in the first τ_i positions of Σ. By analogous reasoning, we have

$$\mathbb{P}(A_j)\mathbb{P}(T_1 \leq \tau_i|\Sigma^-) \leq \mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-] \leq \mathbb{P}(T_1 \leq \tau_i+\tau_j|\Sigma^-)-\mathbb{P}(A_i)\mathbb{P}(\tau_i < T_1 \leq \tau_i+\tau_j|\Sigma^-), \tag{46}$$

where the right inequality follows because in order for group G_j to get utility one, we must have $T_1 \leq \tau_i + \tau_j$, and if $T_1 \in (\tau_i, \tau_i + \tau_j]$, then a member of G_j must appear before all members of G_i.

We now prove the upper-bound in (44). Combining (45) and (46), we see that

$$\begin{aligned}
\frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-]} &\leq \frac{\mathbb{P}(T_1 \leq \tau_i + \tau_j|\Sigma^-) - \mathbb{P}(A_i)\mathbb{P}(\tau_i < T_1 \leq \tau_i + \tau_j|\Sigma^-)}{\mathbb{P}(A_i)\mathbb{P}(T_1 \leq \tau_i|\Sigma^-)} \\
&= \frac{\mathbb{P}(T_1 \leq \tau_i + \tau_j|\Sigma^-)(1-\mathbb{P}(A_i)) + \mathbb{P}(A_i)\mathbb{P}(T_1 \leq \tau_i|\Sigma^-)}{\mathbb{P}(A_i)\mathbb{P}(T_1 \leq \tau_i|\Sigma^-)} \\
&= \frac{|G_j|}{|G_i|}\frac{\mathbb{P}(T_1 \leq \tau_i + \tau_j|\Sigma^-)}{\mathbb{P}(T_1 \leq \tau_i|\Sigma^-)} + 1 \\
&\leq \frac{|G_j| - |G_j|^2}{|G_i|} + |G_j| + 1 \\
&\leq \ell.
\end{aligned} \tag{47}$$

The second inequality uses (41). The final inequality follows because if $|G_j| = \ell$, then $|G_i| = \ell$ by (37), and thus the expression is equal to 2;[7] if $|G_j| < \ell$, then the expression is at most ℓ because $\frac{|G_j|-|G_j|^2}{|G_i|} \leq 0$.

Meanwhile, (45) and (46) also imply that

$$\frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-]} \geq \mathbb{P}(A_j) = \frac{|G_j|}{|G_i| + |G_j|}. \tag{48}$$

If $|G_i| < \ell$ or $|G_j| > 1$, the ratio on the right is at least $1/\ell$, and the proof is complete. Thus, all that remains is to show that the lower bound in (44) holds when $|G_i| = \ell$ and $|G_j| = 1$.

Our analysis will condition on both Σ^- and P. We note that

$$\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-, P] > 0 \Leftrightarrow T_1(P) \leq \tau_i(\Sigma^-).$$

Therefore,

$$\frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-]} = \frac{\mathbb{E}_P[\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-, P]]}{\mathbb{E}_P[\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-, P]]} \geq \min_{P:\ T_1(P)\leq\tau_i(\Sigma^-)} \frac{\mathbb{E}[u_j(x(\mathbf{a},\Sigma))|\Sigma^-, P]}{\mathbb{E}[u_i(x(\mathbf{a},\Sigma))|\Sigma^-, P]}.$$

[7]We assume $\ell \geq 2$ because if $\ell = 1$ and all groups have size one, the individual lottery simply selects k agents uniformly at random, and is perfectly fair.

We will show that the quantity on the right is at least $1/\ell$. To do this, we let

$$\tau_{ij}(\Sigma^-) = \tau(k - |G_j| - |G_i| + 1, \Sigma^-), \tag{49}$$

be as defined in 4.6 with size function $|\sigma_t| = a_{\sigma_t}$. If $T_2(P) \leq \tau_{ij}(\Sigma^-)$, then because each agent requests at most $\ell = |G_i|$ tickets, agent j will receive utility of one if first or second in Σ^S. Thus,

$$\frac{\mathbb{E}[u_j(x(\mathbf{a}, \Sigma))|\Sigma^-, P]}{\mathbb{E}[u_i(x(\mathbf{a}, \Sigma))|\Sigma^-, P]} \geq \mathbb{E}[u_j(x(\mathbf{a}, \Sigma))|\Sigma^-, P] \geq \frac{2}{\ell + 1} \geq \frac{1}{\ell}.$$

Meanwhile, if $T_2(P) > \tau_{ij}(\Sigma^-)$, then each group gets utility 1 only if one of its members is first in Σ^S. In this case,

$$\frac{\mathbb{E}[u_j(x(\mathbf{a}, \Sigma))|\Sigma^-, P]}{\mathbb{E}[u_i(x(\mathbf{a}, \Sigma))|\Sigma^-, P]} = \frac{\mathbb{P}(A_j)}{\mathbb{P}(A_i)} = \frac{1}{\ell}.$$

□

B GROUP LOTTERY

B.1 Incentives

Proof of Proposition 4.8. Fix an arbitrary agent $i \in \mathcal{N}$. Let $\mathbf{a}$ be an action profile such that $\mathbf{a}_{G_i}$ is the group request strategy and $\mathbf{a}_{-G_i}$ is arbitrary. Let $\mathbf{a}'$ denote an alternative strategy profile in which $a'_j = a_j$ for $j \notin G_i$. Let V be the set of valid groups according to $\mathbf{a}$, V' the set of valid groups according to $\mathbf{a}'$, and V^- be the set of valid groups not containing any members of G_i:

$$\begin{aligned} V \ &= \{S \subset \mathcal{N} : a_j = S \ \forall j \in S\}. \\ V' &= \{S \subset \mathcal{N} : a'_j = S \ \forall j \in S\} \\ V^- &= \{S \subset \mathcal{N} \setminus G_j : a_j = S \ \forall j \in S\}. \end{aligned}$$

Note that $V^- \subseteq V \cap V'$, and that agents in G_i do not influence V^-. We generate uniform random orders Σ and Σ' over V and V' (respectively) using Algorithm 2: we first generate a uniform random order Σ^- over V^-, and then extend this to obtain Σ and Σ'. We will prove that for any realization of Σ^-,

$$\mathbb{E}[u_i(x^{GL}(\mathbf{a}, \Sigma))|\Sigma^-] \geq \mathbb{E}[u_i(x^{GL}(\mathbf{a}', \Sigma'))|\Sigma^-]. \tag{50}$$

Because agents in G_i cannot influence Σ^-, it follows immediately that the unconditional expected utility of agent i is also higher under the group request strategy.

If $\sum_{S \in V^-} |S| \leq k - |G_i|$, then the group request strategy guarantees that all members of G_i will receive a ticket, so there is nothing to prove. Otherwise, let

$$\tau(\Sigma^-) = \tau(k - |G_i| + 1, \Sigma^-),$$

be as in Definition 4.6 where the size function is the cardinality of the valid group declared by the agent, that is, $|\sigma_t| = |a_{\sigma_t}|$. Intuitively, τ is the first point at which the number of remaining tickets would be less than $|G_i|$, when processing valid groups in V^- according to order Σ^-.

Because agents in G_i follow the group request strategy under $\mathbf{a}$, we have $V = V^- \cup G_i$. Members of G_i get a payoff of 1 if and only if G_i is in the first $\tau(\Sigma^-)$ positions of Σ. Therefore,

$$\mathbb{E}[u_i(x^{GL}(\mathbf{a}, \Sigma))|\Sigma^-] = \frac{\tau(\Sigma^-)}{1 + |V^-|}. \tag{51}$$

We now turn to the action profile $\mathbf{a}'$. Because the Group Lottery gives at most one ticket to each agent, i gets a payoff of 1 if and only if all members of G_i receive a ticket. This is not possible unless (i) every agent in G_i is included in a valid group in V', and (ii) in the order Σ', all valid groups in $V' \backslash V^-$ appear before group $S = \Sigma^-_{\tau(\Sigma^-)}$. According to the algorithm, the conditional probability of (ii) given Σ^- is at most

$$\frac{\tau(\Sigma^-)}{2 + |V^-|} \frac{\tau(\Sigma^-) - 1}{1 + |V^-|},$$

which is smaller than the right side of (51), implying that group G_i has not benefited from its deviation.

Next, we show that any other strategy is not dominant. Let $j \notin G_i$ and $\tilde{\mathbf{a}}$ denote an action profile such that $j \in \tilde{a}_i$, $i \notin \tilde{a}_j$ and the remaining actions $\tilde{\mathbf{a}}_{-\{i,j\}}$ are arbitrary. Under $\tilde{\mathbf{a}}$ agent i is not in a valid group then it's not award and group G_i get a payoff of 0. This is strictly less than the payoff under a group request, which is greater than the probability of G_i being the first group to be processed. Therefore, we can restrict to strategies $\hat{\mathbf{a}}$ where $\hat{a}_{i'} \subset G_i$ for any $i' \in G_i$. Furthermore, G_i will have a positive expected payoff only if under $\hat{\mathbf{a}}$ its members are divided into two or more valid groups. Let actions $\hat{\mathbf{a}}_{-G_i}$ be such that $\hat{a}_j = \mathcal{N} \setminus G_i$ for any $j \in \mathcal{N} \setminus G_i$, and $\hat{V}$ be the set of valid groups according to $\hat{a}$,

$$\hat{V} \ = \{S \subset \mathcal{N} : \hat{a}_j = S \ \forall j \in S\}.$$

Observe that $|\hat{V}| \geq 3$. By assumption $n > k$, so G_i will get a payoff of 1 if and only if valid group $\mathcal{N} \setminus G_i$ is the last valid group to be processed. This event occurs with probability

$$\frac{(|\hat{V}|-1)!}{|\hat{V}|!} = \frac{1}{|\hat{V}|}.$$

This is strictly smaller than 1/2 the expected utility when G_i select a group request. □

B.2 Performance

Lemma B.1. *Fix any instance and any pair of agents $i, j \in \mathcal{N}$. Let $\mathbf{a}$ be an action profile under the Group Lottery such that G_i and G_j select the group request strategy. If $|G_i| \geq |G_j|$, then*

$$u_i(\pi^{GL}(\mathbf{a})) \leq u_j(\pi^{GL}(\mathbf{a})). \tag{52}$$

Proof of Lemma B.1. Let V be the set of valid groups given $\mathbf{a}$. Observe that by assumption $G_i, G_j \in V$. We define S_i, S_j to be the set of orders over V that guarantee a payoff of 1 to group G_i and G_j, respectively. It suffices to show that

$$|S_j| \geq |S_i|.$$

To prove this, we will construct an injective map $f : S_i \to S_j$. We let f be the map that only swap the positions of G_i and G_j, keeping all the remaining positions unchanged. Clearly f is injective. Thus, all that remains to show is that for any $\sigma \in S_i$, $f(\sigma) \in S_j$. Fix $\sigma \in S_i$. If $\sigma \in S_j$, then there are enough tickets to satisfy both groups and $f(\sigma) \in S_i \cap S_j \subseteq S_j$. If $\sigma \notin S_j$, then then the number of tickets remaining before G_j is processed under $f(\sigma)$ is the same as the number of tickets remaining before G_i is processed under σ. Because $\sigma \in S_i$, this number is at least $|G_i|$ which by assumption is at least $|G_j|$, so $f(\sigma) \in S_j$. □

Fact 1. *For any $a, b, c \in \mathbb{R}$ such that $a \leq b$ and $c \geq 0$, then*

$$\frac{a}{b} \leq \frac{a+c}{b+c}. \tag{53}$$

Proof of Proposition 4.11. First, we will prove the right inequality of (19). If $k \geq \sum_i a_i - \bar{a} + 1$, then our upper bound is at least n and immediately holds. Hence, without loss of generality we can assume $k \leq \sum_i a_i - \bar{a}$. We let $S_t = S_t(\Sigma)$ be the sum of the first t numbers according to Σ, that is,

$$S_t = \sum_{i=1}^{t} a_{\Sigma_i}. \tag{54}$$

We define

$$Z_t^* = \frac{S_t - t\mu}{n-t}. \tag{55}$$

As mentioned in [10], the sequence $Z_1^*, \ldots, Z_{n-1}^*$ is a forward martingale. Furthermore, $k \leq \sum_i a_i - \bar{a}$ implies $\mathbb{P}(\tau \leq n-1) = 1$ so τ is bounded and Z_τ^* is well defined. Hence, we can apply Doob's optional stopping theorem to obtain

$$\mathbb{E}\left[\frac{S_\tau - \tau\mu}{n-\tau}\right] = \mathbb{E}[Z_\tau^*] = \mathbb{E}[Z_1^*] = 0. \tag{56}$$

From the definition of τ, we get

$$\mathbb{E}\left[\frac{S_\tau}{n-\tau}\right] \leq (k + \bar{a} - 1)\mathbb{E}\left[\frac{1}{n-\tau}\right]. \tag{57}$$

We claim that

$$\mathbb{E}[\tau]\mathbb{E}\left[\frac{1}{n-\tau}\right] \leq \mathbb{E}\left[\frac{\tau}{n-\tau}\right]. \tag{58}$$

For any $x < n$, define

$$f(x) = (x - \mathbb{E}[\tau])\left(\frac{1}{n-x} - \frac{1}{n - \mathbb{E}[\tau]}\right).$$

Note that $f(x) \geq 0$ for all x, so $\mathbb{E}[f(\tau)] \geq 0$. Thus,

$$0 \leq \mathbb{E}[f(\tau)] = \mathbb{E}\left[(\tau - \mathbb{E}[\tau])\left(\frac{1}{n-\tau} - \frac{1}{n - \mathbb{E}[\tau]}\right)\right] = \mathbb{E}\left[\frac{\tau - \mathbb{E}[\tau]}{n-\tau}\right].$$

Combining equations (56), (57) and (58) yields our desired result.

Now, we prove the left inequality of (19). If $k \leq \bar{a}$, then our lower bound is at most 1 and immediately holds, then without loss of generality we can assume $k \geq \bar{a} + 1$. To construct Σ we will generate a random order Σ' and iterate through it backwards, that is, $\Sigma_t = \Sigma'_{n-t+1}$ for $t = 1, \ldots, n$. We claim that for every Σ,

$$\tau(k, \Sigma) + \tau(\sum_i a_i - k + 1, \Sigma') = n + 1. \tag{59}$$

It suffices to show that

$$\sum_{t=1}^{\tau(\sum_i a_i - k + 1, \Sigma')} a_{\Sigma'_t} = \sum_{t=\tau(k,\Sigma)}^{n} a_{\Sigma_t}. \tag{60}$$

From the definition of τ, we have that

$$\sum_{t=\tau(k,\Sigma)+1}^{n} a_{\Sigma_t} = \sum_{i=1} a_i - \sum_{t=1}^{\tau(k,\Sigma)} a_{\Sigma_t} \leq \sum_{i=1} a_i - k.$$

Similarly,

$$\sum_{t=\tau(k,\Sigma)}^{n} a_{\Sigma_t} = \sum_{i=1} a_i - \sum_{t=1}^{\tau(k,\Sigma)-1} a_{\Sigma_t} \geq \sum_{i=1} a_i - k + 1.$$

Applying the upper bound in (19) to (59) immediately implies

$$\mathbb{E}[\tau(k, \Sigma)] = n + 1 - \mathbb{E}[\tau(\sum_i a_i - k + 1, \Sigma')] \geq n + 1 - \frac{\sum_i a_i - k + \bar{a}}{\mu} = 1 + \frac{k - \bar{a}}{\mu}.$$

We now turn to equation (20)[8]. For any order σ, we define $h(\sigma)$ to be the order that: (i) is identical to σ from position $\tau(k + k', \sigma)$ until the end, and (ii) flip the ordering of all elements from position 1 to $\tau(k + k', \sigma) - 1$. More precisely,

$$h(\sigma)_t = \begin{cases} \sigma_t & \text{if } t \geq \tau(k + k', \sigma), \\ \sigma_{\tau(k+k',\sigma)-t} & \text{if } t < \tau(k + k', \sigma). \end{cases} \tag{61}$$

We claim that

$$\tau(k + k', \sigma) = \tau(k + k', h(\sigma)).$$

This implies that $h(h(\sigma))) = \sigma$, which further implies that h is a bijective map. This follows from 2 observations: (i) the elements in the first $\tau(k + k', \sigma) - 1$ positions are the same in both orders, and do not sum to $k + k'$. Additionally, (ii) the agents in the first $\tau(k + k', \sigma)$ positions are also the same in both orders, and they do sum to $k + k'$.

We will show that for any $\sigma \in \mathcal{O}_{[n]}$,

$$\tau(k, \sigma) + \tau(k', h(\sigma)) \geq \tau(k + k', \sigma). \tag{62}$$

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[8]We thank Matt Weinberg for suggesting this proof.

This implies our result as

$$\begin{aligned}
\mathbb{E}[\tau(k,\Sigma)+\tau(k',h(\Sigma))] &= \frac{1}{n!}\left(\sum_{\sigma\in\mathcal{O}_{[n]}}\tau(k,\sigma)+\tau(k',\sigma)\right)\\
&= \frac{1}{n!}\left(\sum_{\sigma\in\mathcal{O}_{[n]}}\tau(k,\sigma)+\tau(k',h(\sigma))\right)\\
&\geq \frac{1}{n!}\left(\sum_{\sigma\in\mathcal{O}_{[n]}}\tau(k+k',\sigma)\right)\\
&= \mathbb{E}[\tau(k+k',\Sigma)].
\end{aligned}$$

The second equality follows as h is a bijective map, and the inequality follows from (62). Thus all that remains is to show (62). From the definition of τ, we have

$$\sum_{t=1}^{\tau(k,\sigma)} a_{\sigma_t} \geq k,$$

$$\sum_{t=1}^{\tau(k+k',\sigma)-1} a_{\sigma_t} < k+k'.$$

Implying that

$$\sum_{t=\tau(k,\sigma)+1}^{\tau(k+k',\sigma)-1} a_{\sigma_t} < k'. \tag{63}$$

Moreover, from the definition of h in (61) it follows that

$$\{h(\sigma)_1,\ldots,h(\sigma)_{\tau(k+k',\sigma)-\tau(k,\sigma)-1}\} = \{\sigma_{\tau(k+k',\sigma)-1},\ldots,\sigma_{\tau(k,\sigma)+1}\}. \tag{64}$$

Combining (63) and (64) yields

$$\sum_{t=1}^{\tau(k+k',\sigma)-\tau(k,\sigma)-1} a_{h(\sigma)_t} < k'.$$

This implies (62) as by definition τ is integral and

$$\tau(k',h(\sigma)) > \tau(k+k',\sigma)-\tau(k,\sigma)-1.$$

$\square$

Proof of Lemma 4.10. Because all groups are playing the group request, the set of valid groups is $\mathcal{G}$. In what follows we fix an arbitrary agent $i\in\mathcal{N}$. We construct a random order Σ over $\mathcal{G}$ using Algorithm 2: we generate an order Σ^{-i} over $\mathcal{G}\setminus G_i$ and then extend it to $\mathcal{G}$. By Proposition A.1, the resulting order Σ is uniformly distributed. We let $\tau^{-i}=\tau(k-|G_i|+1,\Sigma^{-i})$ be the number of positions in Σ that ensure G_i a payoff of 1 given Σ^{-i}. Note that τ^{-i} is well defined as $k<n$ implies $k-|G_i|+1\leq n-|G_i|=\sum_t|\Sigma_t^{-i}|$. Moreover, if G_i is in the first τ^{-i} positions of Σ, then it gets a payoff of 1 as the number of remaining tickets before it is processed is at least

$$k-\sum_{t=1}^{\tau^{-i}-1}|\Sigma_t^{-i}| \geq k-(k-|G_i|)=|G_i|.$$

On the other hand, if G_i is in the last $m - \tau^{-i}$ positions of Σ, then it gets a payoff of 0 because the number of remaining tickets when G_i is processed is at most

$$k - \sum_{t=1}^{\tau^{-i}} |\Sigma_t^{-i}| \le k - (k - |G_i| + 1) = |G_i| - 1.$$

Therefore,

$$u_i(x^{GL}(\mathbf{a}, \Sigma)) = \mathbb{E}[\mathbb{E}[u_i(x^{GL}(\mathbf{a}, \Sigma))|\Sigma^{-i}]] = \frac{\mathbb{E}[\tau^{-i}]}{m}. \tag{65}$$

By Proposition 4.11 equation (19), we have

$$\mathbb{E}[\tau^{-i}] \le \frac{k - G_i + \max_G |G|}{(n - |G_i|)/(m-1)}. \tag{66}$$

This and equation (65) yields

$$u_i(\pi^{GL}(\mathbf{a})) \le \left(\frac{k - G_i + \max_G |G|}{n - |G_i|}\right)\left(\frac{m-1}{m}\right). \tag{67}$$

Because there is a group of size $|G_i|$ and the remaining $n - |G_i|$ agents can be in at most $n - |G_i|$ groups of size 1, we have

$$m \le n - |G_i| + 1. \tag{68}$$

This implies

$$\frac{m-1}{m} \le \frac{n - |G_i|}{n - |G_i| + 1}. \tag{69}$$

From (67) and (69) it follows that

$$u_i(\pi^{GL}(\mathbf{a})) \le \frac{k - G_i + \max_G |G|}{n - |G_i| + 1}. \tag{70}$$

Applying Fact 1 with $a = k + \max_G |G| - |G_i|$, $b = n + 1 - |G_i|$ and $c = |G_i| - 1$, we get

$$\frac{k + \max_G |G| - |G_i|}{n - |G_i| + 1} \le \frac{k + \max_G |G| - 1}{n} \le \frac{k}{n}(1 + \kappa). \tag{71}$$

Note that to apply Fact 1 we need $a \le b$, we can assume this without loss of generality. Otherwise, $a > b$ or equivalently

$$k + \max_G |G| > n + 1. \tag{72}$$

We claim that if the inequality above holds then $\frac{k}{n}(1 + \kappa) > 1$, hence, our bound is vacuous. This can be seen by noting that

$$1 + \kappa \ge \frac{k + \max_G |G| - 1}{k} > \frac{n}{k}. \tag{73}$$

The last inequality follows by (72). □

Proof of Theorem 4.9. Fix $\alpha, \kappa \in (0, 1)$ and an arbitrary instance in $I(\kappa, \alpha)$. In what follows, we let $\mathbf{a}$ denote the group request action profile under the Group Lottery. We define s to be the maximum group size, that is, $s = \max_{G \in \mathcal{G}} |G|$.

We begin with the efficiency guarantee. For any order $\sigma \in O_{\mathcal{G}}$,

$$U(x^{GL}(\mathbf{a}, \sigma)) \ge 1 - \frac{s-1}{k} \ge 1 - \kappa, \tag{74}$$

where the second inequality follows as our instance is in $I(\kappa, \alpha)$. This is fairly trivial: if we let

$$\tau(\sigma) = \tau(k + 1, \sigma),$$

be as in Definition 4.6 where the size of a group is its number of elements. Then $U(x^{GL}(\mathbf{a}, \sigma))$ is exactly $\frac{1}{k}\sum_{j=1}^{\tau(\sigma,k+1)-1}|\sigma_j|$, which is at least $\frac{1}{k}(k-(s-1))$, because adding one more group (of size at most s) brings the sum above k. From (74), it immediately follows that if Σ is a random order on $\mathcal{G}$, then

$$\mathbb{E}[U(x^{GL}(\mathbf{a}, \Sigma))] \geq 1-\kappa.$$

We now show that in this setting the outcome is $(1-2\kappa)$-fair. Our goal is to show that for any pair of agents $i, j \in \mathcal{N}$,

$$\frac{u_i(\pi^{GL}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))} \geq 1-2\kappa. \tag{75}$$

By Lemma B.1, we can assume without loss of generality that

$$|G_i| = \max_{G\in\mathcal{G}}|G| \text{ and } |G_j| = \min_{G\in\mathcal{G}}|G|. \tag{76}$$

We let μ^{-i} be the average group size in $\mathcal{G}\setminus G_i$, more precisely,

$$\mu^{-i} = \frac{n-|G_i|}{m-1}. \tag{77}$$

We claim that

$$\frac{u_i(\pi^{GL}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))} \geq \frac{k-2|G_i|+1+\mu^{-i}}{k-|G_j|+\mu^{-i}}. \tag{78}$$

This implies our result as

$$\frac{k-2|G_i|+1+\mu^{-i}}{k-|G_j|+\mu^{-i}} \geq \frac{k-2|G_i|+1+|G_j|}{k} \geq 1-\frac{2(|G_i|-1)}{k} \geq 1-2\kappa.$$

In the first inequality, we apply Fact 1 with $a = k-2|G_i|+1+|G_j|$, $b = k$ and $c = \mu^{-i}-|G_j|$. The last inequality follows from the definition of $I(\kappa, \alpha)$ in (8). Remember that for any instance in $I(\kappa, \alpha)$, we have

$$\frac{|G_i|-1}{k} = \frac{\max_{G\in\mathcal{G}}|G|-1}{k} \leq \kappa.$$

We now turn to the proof of equation (78). Let Σ^{-i} be a uniform order on $\mathcal{G}\setminus G_i$. Applying Lemma 4.10 to agents i, j, it follows that

$$\frac{u_i(x^{GL}(\mathbf{a}, \Sigma))}{u_j(x^{GL}(\mathbf{a}, \Sigma))} = \frac{\mathbb{E}[\tau(k-|G_i|+1, \Sigma^{-i})]}{\mathbb{E}[\tau(k-|G_j|+1, \Sigma^{-j})]}. \tag{79}$$

By Proposition 4.11 equation (20), we have

$$\mathbb{E}[\tau(k-|G_j|+1, \Sigma^{-j})] \leq \mathbb{E}[\tau(k-|G_i|+1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i|-|G_j|, \Sigma^{-j})]. \tag{80}$$

We claim that for any constant $c \in \mathbb{N}$, such that $c \leq n-|G_i| = \sum_t |\Sigma_t^{-i}|$,

$$\mathbb{E}[\tau(c, \Sigma^{-j})] \leq \mathbb{E}[\tau(c, \Sigma^{-i})]. \tag{81}$$

We now show (81). We generate Σ^{-i} using Σ^{-j} in the following way:

$$\Sigma_t^{-i}(\Sigma^{-j}) = \begin{cases} \Sigma_t^{-j} & \text{if } \Sigma_t^{-j} \neq G_i, \\ G_j & \text{otherwise.} \end{cases}$$

Note that by construction for any Σ^{-j}, $\tau(c, \Sigma^{-i}(\Sigma^{-j})) \geq \tau(c, \Sigma^{-j})$. This establishes (81). Applying (81) twice we get

$$\mathbb{E}[\tau(k-|G_i|+1, \Sigma^{-j})] + \mathbb{E}[\tau(|G_i|-|G_j|, \Sigma^{-j})] \leq \mathbb{E}[\tau(k-|G_i|+1, \Sigma^{-i})] + \mathbb{E}[\tau(|G_i|-|G_j|, \Sigma^{-i})]. \tag{82}$$

Then by (80) and (82),

$$\frac{\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})]}{\mathbb{E}[\tau(k-|G_j|+1,\Sigma^{-j})]} \geq \frac{\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})]}{\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})]+\mathbb{E}[\tau(|G_i|-|G_j|,\Sigma^{-i})]}. \tag{83}$$

By Proposition 4.11 equation (19), we have

$$\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})] \geq \frac{k-2|G_i|+1+\mu^{-i}}{\mu^{-i}}, \tag{84}$$

$$\mathbb{E}[\tau(|G_i|-|G_j|,\Sigma^{-i})] \leq \frac{2|G_i|-|G_j|-1}{\mu^{-i}}. \tag{85}$$

From (84) and (85), we have

$$\begin{aligned}\frac{\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})]}{\mathbb{E}[\tau(k-|G_i|+1,\Sigma^{-i})]+\mathbb{E}[\tau(|G_i|-|G_j|,\Sigma^{-i})]} &\geq \frac{k-2|G_i|+1+\mu^{-i}}{k-2|G_i|+1+\mu^{-i}+2|G_i|-|G_j|-1}\\ &= \frac{k-2|G_i|+1+\mu^{-i}}{k-|G_j|+\mu^{-i}}.\end{aligned}$$

□

B.2.1 Tightness.

Proposition B.2. *For any $\beta,\epsilon>0$, there exists $\alpha,\kappa\in(0,1)$ and an instance in $I(\kappa,\alpha)$ such that no random allocation is β-fair and $1-(1-\epsilon)\kappa$ efficient.*

Proof. Fix $\epsilon>0$ and let $m,s,r\in\mathbb{N}$. We consider an instance with $k=rs-1$ tickets, with one group of size $s-1$ and $m-1$ groups of size s. Let i be a member of the group of size $s-1$. Let π be any random allocation. Because at most $r-1$ of the large groups can be satisfied in any deterministic allocation, we have that

$$s(m-1)\min_{j\notin G_i}\{u_j(\pi)\} \leq \sum_{j\notin G_i} u_j(\pi) \leq s(r-1). \tag{86}$$

If the allocation is β-fair, then it must be the case that

$$\beta u_i(\pi) \leq \min_{j\notin G_i}\{u_j(\pi)\} \leq \frac{r-1}{m-1}, \tag{87}$$

where the second inequality follows from (86).

Observe that if G_i is successful then there will be no tickets wasted; otherwise, there will be $s-1$ tickets wasted. Hence, the utilization under π is

$$1-(1-u_i(\pi))\,\frac{s-1}{k} \leq 1-\left(1-\frac{r-1}{\beta(m-1)}\right)\frac{s-1}{k}. \tag{88}$$

If we choose r,m such that $(r-1)/(m-1)<\epsilon\beta$, and define $\kappa=(s-1)/k$ and $\alpha=k/n$, then our instance is in $I(\kappa,\alpha)$, and (88) implies that utilization is strictly smaller than $1-(1-\epsilon)\kappa$. □

Proposition B.3. *For any $\epsilon>0$, there exists $\alpha,\kappa\in(0,1)$ and an instance in $I(\kappa,\alpha)$ such that the group request equilibrium outcome of the Group Lottery is not $1-(2-\epsilon)\kappa$ fair.*

Proof. Fix $\epsilon>0$. We will show that for some $\alpha,\kappa\in(0,1)$, we can construct an instance in $I(\kappa,\alpha)$ such that the group request equilibrium outcome of the Group Lottery is not $1-(2-\epsilon)\kappa$ fair.

If $\epsilon>1$, then we can use a simple example: consider an instance with 3 tickets, one group of size 1 and $m-1$ groups of size 2. In this example, the small group gets a payoff of 1 with probability $\frac{2}{m}$. The large group gets a payoff of 1 with probability $\frac{1}{m}(1+\frac{1}{m-1})$. If we take the limit as m goes to infinity, the ratio of expected utility of the small to large group converges to $1/2$. Moreover, this instance is in $I(\kappa, \alpha)$ for every $\kappa \in [1/3, 1)$ and $\alpha \in (0, 1)$. Therefore, we will assume without loss of generality that

$$\epsilon \leq 1. \tag{89}$$

Let $m, s, r \in \mathbb{N}$ be such that

$$s \geq 4, \; \frac{2}{r+1} < \epsilon \text{ and } m \geq r+2. \tag{90}$$

We consider an instance with $k = r(s-2)+1$ tickets. Let there be one group of size 1, one group of size s and $m-2$ groups of size $s-2$. Now, we choose $\kappa, \alpha \in (0,1)$ in a way that ensures our instance to be in $I(\kappa, \alpha)$:

$$\frac{r+1}{m} = \alpha, \tag{91}$$

$$\delta < \frac{\frac{\epsilon}{r} - \frac{2}{r(r+1)}}{2-\epsilon}, \tag{92}$$

$$\frac{1}{r} + \delta = \kappa. \tag{93}$$

First, we will see that κ, α are in $(0,1)$. For α, this follows immediately from the last condition over r and m in (90) and (91). For κ, we use that (89), (90) and (92) combined imply $\delta < \frac{1}{r} \leq \frac{1}{2}$, thus from (93) we have

$$\kappa = \frac{1}{r} + \delta < \frac{2}{r} \leq 1.$$

Second, we verify that our instance is indeed in $I(\kappa, \alpha)$:

$$\frac{k}{n} = \frac{r(s-2)+1}{(m-1)(s-2)+3} \leq \frac{r+1}{m} = \alpha.$$

The first inequality follows from Fact 1 using $a = r(s-2)+1, b = (m-1)(s-2)+1$ and $c = s-3$. The last equality follows from (91). Besides,

$$\lim_{s\to\infty} \frac{s-1}{k} = \lim_{s\to\infty} \frac{s-1}{r(s-2)+1} = \frac{1}{r}.$$

Therefore, there exists $\bar{s}(\delta) \in \mathbb{N}$ such that for $s \geq \bar{s}$,

$$\frac{s-1}{r(s-2)+1} \leq \frac{1}{r} + \frac{\delta}{2} = \kappa - \delta + \frac{\delta}{2} < \kappa.$$

The equality follows from (93).

In this setting, the large group gets a payoff of 1 if and only if it's behind at most $r-2$ of the medium-sized groups; this occurs with probability $\frac{r-1}{m-1}$. The small group gets a payoff of 1 if it's behind at most r of the medium-sized groups and ahead of the large group. This occurs with probability at least $\frac{r+1}{m-1}\left(1 - \frac{r+1}{m}\right)$. If we take the limit as m goes to infinity, the ratio of expected utility of the small to large group converges to

$$\frac{r-1}{r+1} = 1 - \frac{2}{r+1}.$$

Thus, it suffices to show that

$$\frac{2}{r+1} > (2-\epsilon)\kappa. \tag{94}$$

Observe that

$$\frac{2}{r+1} = \frac{2}{r} - \frac{2}{r(r+1)}. \tag{95}$$

From equations (92) and (93), it follows that

$$(2-\epsilon)\kappa = (2-\epsilon)\left(\frac{1}{r}+\delta\right) = \frac{2-\epsilon}{r} + (2-\epsilon)\delta < \frac{2-\epsilon}{r} + \frac{\epsilon}{r} - \frac{2}{r(r+1)} = \frac{2}{r} - \frac{2}{r(r+1)}. \tag{96}$$

Combining (95) and (96) yields (94). □

C WEIGHTED INDIVIDUAL LOTTERY

C.1 Incentives

Proposition C.1. *Algorithm 1 generates a random order* $\Sigma \in \mathcal{O}_{\mathcal{N}}$ *distributed according to* (21) *conditional on* $\mathbf{a}$.

Proof of Proposition C.1. Fix any order σ over $\mathcal{N}$. Let $Y_j = a_j X_j$. It follows that $\mathbb{P}(Y_j > t) = e^{-t/a_j}$, so each Y_j is distributed as an exponential random variable with mean a_i. Moreover, the Y_j are independent. Let Σ be the order generated by the algorithm. We have that

$$\mathbb{P}(\Sigma_1 = j) = \mathbb{P}(Y_j = \min_{i\in\mathcal{N}} Y_i) = \frac{1/a_j}{\sum_{i\in\mathcal{N}} 1/a_i},$$

where the second equality follows from well-known properties of the minimum of exponential random variables.[9] Furthermore, the definition of Σ and the memoryless property of exponential random variables imply that for $t \in \{1,\ldots,n\}$ and $j \notin \Sigma_{[t-1]}$,

$$\mathbb{P}(\Sigma_t = j|\Sigma_{[t-1]}) = \mathbb{P}(Y_j = \min_{i\in\mathcal{N}\setminus\Sigma_{[t-1]}} Y_i) = \frac{1/a_j}{\sum_{i\in\mathcal{N}\setminus\Sigma_{[t-1]}} 1/a_i}.$$

This implies that

$$\Pr(\Sigma = \sigma) = \prod_{t=1}^{n} \mathbb{P}(\Sigma_t = \sigma_t|\Sigma_{[t-1]} = \sigma_{[t-1]}) = \prod_{t=1}^{n} \frac{1/a_{\sigma_t}}{\sum\limits_{i\in\mathcal{N}\setminus\sigma_{[t-1]}} 1/a_i},$$

as claimed. □

Proof of Proposition 4.12. We start by proving that agents have no incentives to request more tickets than their group size. Formally, if we let $i \in G$, $a_i = |G|$ and $a_i' > |G|$ then for every action profile $\mathbf{a}_{-i} \in A_{-i}$,

$$u_i(\pi^{SPL}(a_i, \mathbf{a}_{-i})) \geq u_i(\pi^{SPL}(a_i', \mathbf{a}_{-i})).$$

This follows because the set of orders over agents in which G get a payoff of 1 is the same under both strategies, and by reducing its request agent i improves her probability of being drawn early.

We now show that if group G is such that $|G| \leq 3$, then selecting group request $\mathbf{a}_G$ is dominant for G. Given an action profile $\mathbf{a} \in A$, we generate a random order over agents Σ using the Algorithm 1: we draw iid exponential random variables X_i for each agent i, and sort agents in increasing order according to $a_i X_i$. From Proposition C.1, it follows that Σ is distributed according to (21) conditional on $\mathbf{a}$. Let T be as in Definition (22), intuitively T is the score threshold that some members of G must clear in order to ensure the group a payoff of 1. Furthermore, when G is selecting the group request strategy, it will get a payoff of 1 if and only if at least one of its members has a score lower than T, that is, $\min_{i\in G}\{a_i X_i\} < T$. Because $\min_{i\in G}\{a_i X_i\} \sim Exp(1)$, it follows that for $i \in G$,

$$\mathbb{E}[u_i(\pi^{SPL}(\mathbf{a}_G, \mathbf{a}_{-G}))|T] = \mathbb{P}(\min_{i\in G}\{a_i X_i\} < T) = 1 - e^{-T}. \tag{97}$$

Because T is independent of the strategy followed by G, it suffices to show that for any deviation $\mathbf{a}_G'$ the conditional expected utility of G given T is less than or equal the right side of (97).

[9]See e.g. https://en.wikipedia.org/wiki/Exponential_distribution.

We have already established that it is never beneficial for agents to request more tickets than their group size. Hence, without loss of generality we assume that each member of G will request at most $|G|$ tickets.

If $|G| = 1$, then the group request is the only feasible strategy so it's dominant.

If $|G| = 2$, then the only deviations we need to consider are $\mathbf{a}'_G = (1, 2), (1, 1)$. The first strategy is dominated by the group request, because the allocation of the member requesting 1 ticket is irrelevant for the outcome of group G. Under the second strategy, G gets a payoff of 1 if and only if both members have a score lower than T. In particular, agent $i \in G$ must have a score lower than T. This happens with probability

$$\mathbb{P}(a'_i X_i < T) = \mathbb{P}(X_i < T) = 1 - e^{-T}.$$

Note that the quantity above coincides with (97), implying that the utility of G when selecting $\mathbf{a}'_G = (1, 1)$ is at most its utility under the group request strategy.

If $|G| = 3$, there are 27 feasible strategies (26 deviations from the group request), but by symmetry we only need to evaluate 9 of them:

$$\mathbf{a}'_G = (1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 3, 3), (2, 2, 2), (2, 2, 3), (2, 3, 3).$$

We argue now that the group request dominates all strategies above in which there is at least one agent requesting 1 ticket. Note that under any of these strategies, G will get a payoff of 1 only if the remaining 2 members are awarded two or more tickets. From the case $|G| = 2$, we know that the probability of this event is at most the right hand side of (97). This implies that the group request strategy dominates all these deviations.

There are only 3 strategies remaining: $\mathbf{a}'_G = (3, 3, 2), (3, 2, 2), (2, 2, 2)$. The first strategy is dominated by the group request, because the allocation of the member requesting 2 tickets is irrelevant for the outcome of group G. The second strategy is dominated by $(3, 2, 1)$. This follows because the set of orders over agents in which G get a payoff of 1 is the same under both strategies, and by reducing its request the last agent improves her probability of being drawn early. A similar argument shows that the last strategy is dominated by $(1, 2, 2)$. □

Analysis of Example 4.13. Let i be a member of the large group. We let $\mathbf{a} = (\mathbf{a}_{G_i}, \mathbf{a}_{-G_i})$ denote the group request action profile, and $\mathbf{a}'_{G_i}$ denote the strategy where all members of G_i request 2 tickets. We will show that for $n \geq 17$,

$$u_i(\pi^{SPL}(\mathbf{a}'_{G_i}, \mathbf{a}_{-G_i})) \geq u_i(\pi^{SPL}(\mathbf{a}_{G_i}, \mathbf{a}_{-G_i})).$$

Let $m = n - 3$ be the number of groups. We claim that

$$u_i(\pi^{SPL}(\mathbf{a}_{G_i}, \mathbf{a}_{-G_i})) = 1 - \frac{1}{m}, \tag{98}$$

$$u_i(\pi^{SPL}(\mathbf{a}'_{G_i}, \mathbf{a}_{-G_i})) = 1 - \sum_{t=1}^{m} \left(\prod_{i=1}^{t-1} \frac{m-i}{m+2-i} \right) \left(\frac{2}{m+2-t} \right) \left(\prod_{i=t}^{m-1} \frac{m-i}{m+3/2-i} \right). \tag{99}$$

This implies our result as for $m \geq 14$ the expression in (99) exceeds the expression in (98).

First, we will show (98). Because G_i is selecting the group request strategy, it will get a payoff of 0 if and only if all agents from small groups are processed before its members. This event happens with probability

$$\prod_{i=1}^{m-1} \frac{m-i}{m+1-i} = \frac{1}{m}.$$

Secondly, we show (99). If all members of G_i are requesting 2 tickets, then G_i will get a payoff of 0 if and only if three of its members are processed after all agents in small groups. Moreover, the probability that at step $t = 1, \ldots, m$ a member of G_i is processed for the first time is

$$\left(\prod_{i=1}^{t-1} \frac{m-i}{m+2-i}\right)\left(\frac{2}{m+2-t}\right). \tag{100}$$

In the expression above we used that $\sum_{i \in G} 1/a_i' = 2$, and that at the beginning of step $i \leq t$, there are $m + 2 - i$ agents in small groups that have not been processed yet. Note that if $t = 1$, then the expression above reduces to the probability of processing a member of the large group at the first step, that is, $2/(m + 1)$.

Finally, the probability that all the remaining agents in small groups are processed before the three remaining members of the large group is

$$\prod_{i=t}^{m-1} \frac{m-i}{m+3/2-i}. \tag{101}$$

Here we are using that if $j \in G$ was processed at step t, then $\sum_{i \in G \setminus \{j\}} 1/a_i' = 3/2$. Note that if $t = m$, then the expression above is 1.

Multiplying (100) by (101) and summing all possible values of t yields (99). □

C.1.1 Proof of Proposition 4.15.

LEMMA C.2. *Group G gets a payoff of 1 if and only if*

$$\sum_{i \in G} a_i \mathbf{1}(a_i X_i < T) \geq |G|. \tag{102}$$

PROOF OF LEMMA C.2. First, suppose that (102) holds. From the definition of T in (22), it follows that at most $k - |G|$ tickets are allocated to agents not in G who have a score lower than T. Furthermore, as (102) holds it must be the case that the sum of the requests of agents in G who have a score lower than T is at least $|G|$. Therefore, group G is awarded $|G|$ or more tickets.

Conversely, suppose that (102) does not hold. We will consider two cases:

(i) Only agents with score lower than T are awarded.
(ii) There are agents with score T or higher that are awarded.

Assume first that (i) holds. Then as (102) doesn't hold individuals in G must receive fewer than $|G|$ tickets.

Assume now that (ii) holds. From the definition of T in (22), it follows that individuals not in G must receive strictly more than $k - |G|$ tickets. This implies that individuals in G must receive fewer than $|G|$ tickets. □

LEMMA C.3. *Fix an arbitrary group G. Let $r \in \{0, \ldots, |G| - 1\}$. For every strategy $\mathbf{a}_G \in \mathcal{B}_r$, it follows that*

$$\sum_{i \in G} \frac{1}{a_i} \leq r + 1. \tag{103}$$

PROOF OF LEMMA C.3. For simplicity, we shall assume that $G = \{1, \ldots, s\}$ and $a_1 \leq a_2 \leq \cdots \leq a_s$. Thus, a strategy $\mathbf{a}_G \in \mathbb{N}^s$ is in $\mathcal{B}_r$ if and only if

$$\sum_{i=1}^{r+1} a_i \geq s \text{ and } \sum_{i=s-r+1}^{s} a_i \leq s - 1. \tag{104}$$

Consider the following optimization problem:

$$\begin{array}{ll} \max & \sum_{i=1}^{s} 1/a_i \\ \text{subject to} & \sum_{i=1}^{r+1} a_i \geq s \\ & \sum_{i=s-r+1}^{s} a_i \leq s-1. \\ & 1 \leq a_1 \leq \cdots \leq a_s \end{array} \tag{105}$$

Note that from (104) it follows that every strategy $\mathbf{a}_G \in \mathcal{B}_r$ is a feasible solution for this problem. Therefore, to prove (103) it suffices to show that the optimal value of this problem is at most $r+1$.

We start by proving that an optimal solution $\mathbf{a}_G^*$ of (105) must satisfy

$$a_j^* = s - \sum_{i=1}^{r} a_i^* \text{ for every } j = r+1, \ldots, s. \tag{106}$$

Suppose $\mathbf{a} \in \mathcal{B}_r$ is such that $a_j > s + \sum_{i=1}^{r} a_i$ for some $j \in \{r+1, \ldots, s\}$. If we replace a_j by $a'_j = s - \sum_{i=1}^{r} a_i$ then we increase the objective value as $1/a_j < 1/a'_j$. Moreover, $\mathbf{a}'$ will still be in $\mathcal{B}_r$ as $\sum_{i=1}^{r+1} a'_i = \sum_{i=1}^{r} a_i + a'_j = s$ and $\sum_{i=s-r+1}^{s} a'_i \leq \sum_{i=s-r+1}^{s} a_i \leq s-1$. The last inequality follows as $\mathbf{a} \in \mathcal{B}_r$.

It follows from (106) that we can incorporate in (105) the constraints

$$a_{r+1} = \cdots = a_s = s - \sum_{i=1}^{r} a_i,$$

without decreasing the optimal value. Moreover, if we remove the constraint $\sum_{i=s-r+1}^{s} a_i \leq s-1$ then the optimal value will be higher or the same. By including both modifications we obtain the following relaxation of (105):

$$\begin{array}{ll} \max & \sum_{i=1}^{r} \frac{1}{a_i} + \frac{(s-r)}{a_{r+1}} \\ \text{subject to} & \sum_{i=1}^{r+1} a_i = s \\ & 1 \leq a_1 \leq \cdots \leq a_{r+1} \end{array} \tag{107}$$

Clearly an optimum of (107) exists as the objective function is continuous and the feasible set is non-empty and compact. Moreover, we are maximizing a convex function on a convex set then there exists a globally optimal solution that is an extreme point of the feasible set. The extreme points of the feasible set are $\mathbf{a}^0, \ldots, \mathbf{a}^r$, where

$$1 = a_1^j = \cdots = a_j^j, \quad \frac{s-j}{r+1-j} = a_{j+1}^j = \cdots = a_{r+1}^j.$$

Furthermore, the objective value evaluated at any extreme point is equal to $r+1$. To see this note that objective value at $\mathbf{a}^j$ is

$$j + (s-j)\left(\frac{r+1-j}{s-j}\right) = r+1.$$

Therefore, the optimal value of (107) is $r+1$. Because (107) is a relaxation of (105), it follows that the optimal value of (105) is at most $r+1$. □

Proof of Proposition 4.15. Let $r \in \{0, \ldots, s-1\}$. We formulate the problem of finding the strategy in $\mathcal{B}_r$ that maximizes the expected payoff of G given the threshold T as a programming problem. From Lemma C.2 and since we are considering only strategies in $\mathcal{B}_r$, it follows that group G will get a payoff of 1 if and only if there are $r+1$ or more agents with a score lower than T. For each agent $i \in G$, we let B_i be a random variable that indicates if the score of agent i is lower than

T, more precisely, $B_i = \mathbf{1}(a_i X_i < T)$. Observe that given T and any action a_i, because $X_i \sim Exp(1)$ then $B_i \sim Bernoulli(1 - e^{-T/a_i})$. Hence, our formulation is

$$\begin{array}{ll} \max & \mathbb{P}(\sum_{i \in G} B_i \geq r + 1) \\ \text{subject to} & B_i \sim Bernoulli(1 - e^{-T/a_i}) \quad \forall i \in G \\ & \mathbf{a}_G \in \mathcal{B}_r \end{array} \tag{108}$$

Let Z_i be the Poisson random variable of rate T/a_i. Note that Z_i first-order stochastically dominates B_i. Hence, the following problem is a relaxation of (108).

$$\begin{array}{ll} \max & \mathbb{P}(\sum_{i \in G} Z_i \geq r + 1) \\ \text{subject to} & Z_i \sim Poisson(T/a_i) \quad \forall i \in G \\ & \mathbf{a}_G \in \mathcal{B}_r \end{array} \tag{109}$$

Using that the sum of independent Poisson random variables is Poisson-distributed, we have that $\sum_{i \in G} Z_i \sim Poisson(\sum_i T/a_i)$. Moreover, if $X \sim Poisson(\lambda)$ then [?] state the following bound:

$$\mathbb{P}(X \geq x) \leq 1 - e^{-\lambda/x}, \quad x \geq \lambda. \tag{110}$$

If $T \leq 1$, then Lemma C.3 implies

$$\sum_{i \in G} \frac{T}{a_i} \leq \sum_{i \in G} \frac{1}{a_i} \leq r + 1. \tag{111}$$

Therefore, we can apply (110) to obtain

$$\mathbb{P}(\sum_{i \in G} Z_i \geq r + 1) \leq 1 - e^{-(\sum_i T/a_i)/(r+1)} \leq 1 - e^{-T}. \tag{112}$$

The last inequality follows from Lemma C.3.

From (24) we have that $1 - e^{-T}$ correspond to the utility of G under the group request strategy. This implies our result as the optimal value of the relaxation (109) is at most the utility under the group request strategy. □

C.2 Performance

C.2.1 Proof of Theorem 4.16.

Proof of Theorem 4.16. In this proof, whenever we study a mechanism we assume that the action profile selected $\mathbf{a}$ is its corresponding group request strategy.

We start by proving the efficiency guarantee. From Proposition 4.17, we have that for any instance the utilization under the Weighted Individual Lotteryis at least the utilization under the Group Request with Replacement. This can be seen as

$$U(\pi^{SPL}(\mathbf{a})) = \frac{\sum_{i \in \mathcal{N}} u_i(\pi^{SPL}(\mathbf{a}))}{k} \geq \frac{\sum_{i \in \mathcal{N}} u_i(\pi^{GR}(\mathbf{a}))}{k} = U(\pi^{GR}(\mathbf{a})). \tag{113}$$

The inequality follows from (27). Therefore, it suffices to show that for any instance in $I(\kappa, \alpha)$, the Group Lottery with Replacement is $(1 - \kappa)g(\alpha)$-efficient. This follows immediately by Lemma 4.18:

$$U(\pi^{GR}(\mathbf{a})) = \frac{\sum_{i \in \mathcal{N}} u_i(\pi^{GR}(\mathbf{a}))}{k} \geq \frac{n\left(\frac{k}{n}(1 - \kappa)g(\alpha)\right)}{k} = (1 - \kappa)g(\alpha). \tag{114}$$

Now we turn to the fairness guarantee. From Proposition 4.17 , we have that for any instance and any pair of agents i, j,

$$\frac{u_i(\pi^{SPL}(\mathbf{a}))}{u_j(\pi^{SPL}(\mathbf{a}))} \geq \frac{u_i(\pi^{GR}(\mathbf{a}))}{u_j(\pi^{GL}(\mathbf{a}))}. \tag{115}$$

Moreover, combining Lemma 4.18 and Lemma 4.10 yields

$$\frac{u_i(\pi^{GR}(\mathbf{a}))}{u_i(\pi^{GL}(\mathbf{a}))} \geq \frac{\frac{k}{n}(1-\kappa)g(\alpha)}{\frac{k}{n}(1+\kappa)} \geq (1-\kappa)^2 g(\alpha) \geq (1-2\kappa)g(\alpha). \tag{116}$$

The second inequality follows from the fact that for any $x \geq 0$,

$$\frac{1}{1+x} \geq 1-x.$$

□

C.2.2 *Proof of Lemma 4.17.*

Definition C.4. We let $\mathcal{S}_{\mathcal{N}}$ be the set of finite sequences of agents and draw the random sequence $\Sigma \in \mathcal{S}_{\mathcal{N}}$ by letting Σ_t be iid with

$$\mathbb{P}(\Sigma_t = i) = \frac{1/|G_i|}{\sum_{j\in\mathcal{N}} 1/|G_j|}, \tag{117}$$

and stopping once all agents have been drawn at least once, that is, for each $i \in \mathcal{N}$ there exists t such that $i = \Sigma_t$. This occurs with probability one, implying that this procedure generates a valid distribution over $\mathcal{S}_{\mathcal{N}}$.

Definition C.5. Define $\sigma^{GR} : \mathcal{S}_{\mathcal{N}} \to \mathcal{S}_{\mathcal{G}}$ by

$$\sigma_j^{GR}(\Sigma) = G_{\Sigma_j}. \tag{118}$$

Define $\sigma^{IW} : \mathcal{S}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{N}}$ by

$$T_j^{IW}(\Sigma) = \min\{t \in \mathbb{N} : |\Sigma_{[t]}| = j\}, \tag{119}$$

$$\sigma_j^{IW}(\Sigma) = \Sigma_{T_j^{IW}(\Sigma)}. \tag{120}$$

Note that for each $\Sigma \in \mathcal{S}_{\mathcal{N}}$ and each $t \in \mathbb{N}$, $\sigma_{[t]}^{GR}(\Sigma) \subseteq \mathcal{G}$. Define $\sigma^{GL} : \mathcal{S}_{\mathcal{N}} \to \mathcal{O}_{\mathcal{G}}$ by

$$T_j^{GL}(\Sigma) = \min\{t \in \mathbb{N} : |\sigma_{[t]}^{GR}(\Sigma)| = j\}. \tag{121}$$

$$\sigma_j^{GL}(\Sigma) = \sigma_{T_j^{GL}(\Sigma)}^{GR}(\Sigma). \tag{122}$$

Proposition C.6. *Let $\sigma^{GR}, \sigma^{IW}, \sigma^{GL}$ be as in Definitions C.5. If Σ is drawn according to Definition C.4, then*

- *$\sigma^{GLR}(\Sigma)$ a sequence of k elements in $\mathcal{G}$, where each element is independently and uniformly sampled with replacement from $\mathcal{G}$.*
- *$\sigma^{IW}(\Sigma)$ is an order over $\mathcal{N}$ distributed as in* (21) *given a group request action profile.*
- *$\sigma^{GL}(\Sigma)$ is a uniform order over $\mathcal{G}$.*

Proof. From the definition of $\sigma^{IW}(\Sigma)$, we know that it skip every agent in Σ that has already appeared. Hence, we are sequentially sampling agents without replacement, with probability inversely proportional to the size of its groups. Therefore, it correspond to an order over agents distributed according to (21) when each agent is requesting its group size.

From (117) it follows that for each $G \in \mathcal{G}$ and each t, $\mathbb{P}(\Sigma_t \in G) = 1/|\mathcal{G}|$. That is, the marginal distribution over groups is uniform. It immediately follows from the definition of $\sigma^{GLR}(\Sigma)$ that it's sampling groups uniformly at random with replacement. Moreover, from the definition of $\sigma^{GL}(\Sigma)$ we know that it skip every agent in Σ whose group has already appeared. Therefore, we are sampling groups uniformly at random without replacement, generating a uniform order over $\mathcal{G}$. □

Proof of Lemma 4.17. Let Σ be drawn according to Definition C.4, and $\sigma^{GR}, \sigma^{IW}, \sigma^{GL}$ be as in Definitions C.5. From Proposition C.6 it follows that

$$\begin{aligned} u_i(\pi^{GR}(\mathbf{a})) &= \mathbb{E}[u_i(x^{GL}(\mathbf{a}, \sigma^{GR}(\Sigma)))], \\ u_i(\pi^{SPL}(\mathbf{a})) &= \mathbb{E}[u_i(x^{SPL}(\mathbf{a}, \sigma^{IW}(\Sigma)))], \\ u_i(\pi^{GL}(\mathbf{a})) &= \mathbb{E}[u_i(x^{GL}(\mathbf{a}, \sigma^{GL}(\Sigma)))]. \end{aligned}$$

Therefore, it suffices to show that for any realization of Σ,

$$u_i(x^{GL}(\mathbf{a}, \sigma^{GR}(\Sigma))) \leq u_i(x^{SPL}(\mathbf{a}, \sigma^{IW}(\Sigma))) \leq u_i(x^{GL}(\mathbf{a}, \sigma^{GL}(\Sigma))). \tag{123}$$

Observe that given Σ, each of the utilities above is either 0 or 1. Hence, to prove (123) we will show that: (i) if the utility of i under the Group Lottery with Replacement is 1 then its utility under the Weighted Individual Lottery is also 1, and (ii) if the utility of agent i under the Weighted Individual Lottery is 1 then its utility under the Group Lottery is also 1. Because agents are playing the group request strategy, whenever a group or agent is being processed, it is given a number of tickets equal to the minimum of its group size and the number of remaining tickets.

If the utility of i under the Group Lottery with Replacement is 1, then the number of tickets allocated before G_i is processed is at most $k - |G_i|$. Formally, if we let t be the first *time* at which a member of G_i appears in Σ, then

$$\sum_{j=1}^{t-1} |G_{\Sigma_j}| \leq k - |G_i|. \tag{124}$$

In the left hand side, we use that the sequence of groups $\sigma^{GR}(\Sigma)$ is determined by replacing each agent in Σ by its group. In contrast, in the Weighted Individual Lottery, the order over agents $\sigma^{IW}(\Sigma)$ is constructed by skipping all agents in Σ that have already appeared. Hence, in this mechanism the number of tickets allocated before Σ_t appears in $\sigma^{IW}(\Sigma)$ is the same or lower than the left hand side of (124). This implies that the utility of i under the Weighted Individual Lottery is also 1.

Meanwhile, suppose that the utility of i under the Weighted Individual Lottery is 1, then the number of tickets allocated before Σ_t appears in $\sigma^{IW}(\Sigma)$ is at most $k - |G_i|$. In the Group Lottery, the order over groups $\sigma^{GL}(\Sigma)$ is constructed by replacing each agent in Σ by its group, and skipping all groups that have already appeared. Note that each time $\sigma^{IW}(\Sigma)$ skips an agent Σ_j, $\sigma^{GL}(\Sigma)$ also skips the group G_{Σ_j}. Therefore, the number of tickets allocated before G_{Σ_t} appears in $\sigma^{GL}(\Sigma)$ is the same or lower than the number of tickets allocated before Σ_t appears in $\sigma^{IW}(\Sigma)$. Implying that the utility of agent i under the Group Lottery is also 1. ◻

C.2.3 Proof of Lemma 4.18.

Proposition C.7. *Given a set V of m elements and a natural number k, the following algorithm generates a sequence Σ of k elements where Σ_t is independent and uniformly sampled from V:*

(1) Select an element G of V.
(2) Generate a sequence of k elements Σ^-, where Σ^-_t is independently and uniformly draw from $V \setminus G$.
(3) Generate a sequence of k independent binary random variables X, where $X_t \sim Bernoulli(1/m)$.
(4) For $t = 1, \ldots, k$ set

$$\Sigma_t = \begin{cases} G & \text{if } X_t = 1, \\ \Sigma^-_t & \text{otherwise.} \end{cases}$$

Proof. Observe that Σ_t depends only on X_t and Σ^-_t, hence, for any $t' \neq t$, Σ_t is independent of $\Sigma_{t'}$. Furthermore, for any $G' \in V$, $\mathbb{P}(\Sigma_t = G') = 1/m$. □

Proof of Lemma 4.18. In this proof, we assume that the action profile selected $\mathbf{a}$ is the group request strategy, hence, the set of valid groups is $\mathcal{G}$. Fix an arbitrary group G_i. To generate the sequence $\Sigma \in \mathcal{S}_{\mathcal{G}}$ we use the algorithm from Proposition C.7, that is, generate a sequence Σ^- where Σ^-_t is independently and uniformly sample from $\mathcal{G} \setminus G_i$ and then extend it to Σ. Let $S_n = \sum_{t=1}^n |\Sigma^-_t|$. We let $\tau = \tau(k - |G_i| + 1, \Sigma^-)$ be as defined in 4.6 where the size function is the cardinality of each valid group. Intuitively, τ is the number of positions in Σ that ensures a payoff of 1 to G_i given Σ^-. Note that if G_i is in the first τ positions of Σ, then the number of tickets awarded before it's processed is at most $k - |G_i|$. On the other hand, if it's processed after τ groups this number is at least $k - |G_i| + 1$. Therefore, we get

$$u_i(\pi^{GR}(\mathbf{a})) = \mathbb{E}[\mathbb{E}[u_i(\pi^{GR}(\mathbf{a}))|\Sigma^-]] = \mathbb{E}\left[1 - \left(1 - \frac{1}{m}\right)^{\tau}\right]. \tag{125}$$

Thus, to prove equation (28) it suffices to show

$$\mathbb{E}\left[1 - \left(1 - \frac{1}{m}\right)^{\tau}\right] \geq \frac{k}{n}(1-\kappa)g(\alpha). \tag{126}$$

We let m_j be the number of groups of size j in $\mathcal{G} \setminus G_i$, more precisely,

$$m_j = \sum_{G \in \mathcal{G} \setminus G_i} \mathbf{1}\{|G| = j\}.$$

From this definition, it immediately follows that

$$m - 1 = \sum_{j \geq 1} m_j, \tag{127}$$

$$\sum_{j \geq 1} m_j j = n - |G_i|. \tag{128}$$

Define

$$\phi(\theta) = \mathbb{E}[e^{|\Sigma^-_1|\theta}] = \sum_{j \geq 1} \left(\frac{m_j}{m-1}\right) e^{j\theta}, \tag{129}$$

We let $F = \{F_n\}_{n \in \mathbb{N}}$ be the filtration generated by Σ^-. For any $\theta \in \mathbb{R}$, we define the following martingale w.r.t. F_n:

$$\frac{e^{\theta S_n}}{\phi(\theta)^n}. \tag{130}$$

This expression is adapted with respect to F_n, it's bounded as $|\Sigma^-_i| \leq \max_G |G|$ and, as shown below, it satisfies the martingale property:

$$\mathbb{E}\left[\frac{e^{\theta S_n}}{\phi(\theta)^n}|F_{n-1}\right] = \frac{e^{\theta S_{n-1}}}{\phi(\theta)^{n-1}} \mathbb{E}\left[\frac{e^{\theta|\Sigma^-_n|}}{\phi(\theta)}|F_{n-1}\right] = \frac{e^{\theta S_{n-1}}}{\phi(\theta)^{n-1}} \frac{\mathbb{E}\left[e^{\theta|\Sigma^-_n|}\right]}{\phi(\theta)} = \frac{e^{\theta S_{n-1}}}{\phi(\theta)^{n-1}}.$$

Clearly τ is a stopping time w.r.t. F, moreover, it's almost surely bounded because $|\Sigma^-_i| \geq 1$ implies that $\mathbb{P}(\tau \leq k - |G_i| + 1) = 1$. Applying Doob's optional stopping theorem, we get

$$1 = \mathbb{E}\left[\frac{e^{\theta S_1}}{\phi(\theta)}\right] = \mathbb{E}\left[\frac{e^{\theta S_\tau}}{\phi(\theta)^\tau}\right]. \tag{131}$$

Moreover, if we restrict to $\theta > 0$ and use that the definition of τ implies

$$S_\tau \geq k - |G_i| + 1,$$

we obtain

$$\mathbb{E}\left[\frac{e^{\theta S_\tau}}{\phi(\theta)^\tau}\right] \geq e^{\theta(k-|G_i|+1)}\mathbb{E}\left[\phi(\theta)^{-\tau}\right]. \tag{132}$$

Combining equations (131) and (132) yields

$$e^{-\theta(k-|G_i|+1)} \geq \mathbb{E}\left[\phi(\theta)^{-\tau}\right]. \tag{133}$$

To prove equation (126), we need an upper bound on $\mathbb{E}\left[\left(1-\frac{1}{m}\right)^\tau\right]$. Thus, we let θ^* be the unique solution of

$$\phi(\theta) = \left(1-\frac{1}{m}\right)^{-1} = \frac{m}{m-1}. \tag{134}$$

The existence and uniqueness of θ^* is guaranteed because $\phi(\theta)$ is increasing and continuous, $\phi(0) = 1$ and for $\theta \geq 0,\ \phi(\theta) \geq e^\theta$ hence $\phi(\log(\frac{m}{m-1})) \geq \frac{m}{m-1}$. Then equation (133) evaluates to

$$e^{-\theta^*(k-|G_i|+1)} \geq \mathbb{E}\left[\phi(\theta^*)^{-\tau}\right] = \mathbb{E}\left[\left(1-\frac{1}{m}\right)^\tau\right].$$

This implies

$$\mathbb{E}\left[1-\left(1-\frac{1}{m}\right)^\tau\right] \geq 1-e^{-\theta^*(k-|G_i|+1)} = \theta^*(k-|G_i|+1)g(\theta^*(k-|G_i|+1)). \tag{135}$$

The expression above is an increasing function of θ^*. Hence, if $\theta^* \geq 1/n$ then (126) holds as

$$\theta^*(k-|G_i|+1)g(\theta^*(k-|G_i|+1)) \geq \frac{k-|G_i|+1}{n}g\left(\frac{k-|G_i|+1}{n}\right) \tag{136}$$

$$\geq \frac{k-\max_G|G|+1}{n}g\left(\frac{k-|G_i|+1}{n}\right), \tag{137}$$

and since g is a decreasing function we have

$$g\left(\frac{k-|G_i|+1}{n}\right) \geq g\left(\frac{k}{n}\right) \geq g(\alpha). \tag{138}$$

Thus, we assume $\theta^* < 1/n$. Again, because g is a decreasing function it follows that

$$g(\theta^*(k-|G_i|+1)) \geq g\left(\frac{k-|G_i|+1}{n}\right) \geq g\left(\frac{k}{n}\right) \geq g(\alpha). \tag{139}$$

Therefore, it suffices to show that

$$\theta^*(k-|G_i|+1) \geq \frac{k}{n}(1-\kappa). \tag{140}$$

From the definition of θ^*, we get

$$\frac{m}{m-1} = \phi\left(\theta^*\right) = \sum_{j\geq 1}\frac{m_j e^{j\theta^*}}{m-1} \leq \frac{1}{m-1}\sum_{j\geq 1}\frac{m_j}{1-j\theta^*}. \tag{141}$$

In the inequality we use that for any $x < 1,\ e^x \leq 1/(1-x)$. Observe that

$$j\theta^* < j/n \leq \max_G |G|/n < 1.$$

The first inequality follows by our assumption $\theta^* < 1/n$, the second an third as

$$j \leq \max_G |G| \leq k < n.$$

If we multiple both sides of (141) by $(m-1)$ and subtract $(m-1)$ we obtain

$$1 \leq \sum_{j\geq 1}\frac{m_j}{1-j\theta^*} - (m-1) = \sum_{j\geq 1}\frac{m_j}{1-j\theta^*} - \sum_{j\geq 1} m_j = \sum_{j\geq 1}\frac{m_j j\theta^*}{1-j\theta^*}.$$

The first equality follows from equation (127). Besides,

$$\sum_{j\geq 1} \frac{m_j j\theta^*}{1-j\theta^*} \leq \sum_{j\geq 1} \frac{m_j j\theta^*}{1-\max_G |G|\theta^*} = \frac{(n-|G_i|)\theta^*}{1-\max_G |G|\theta^*}.$$

In the first inequality we use that $j \leq \max_G |G|$. The equality follows from equation (128). Combining both expressions above yields

$$1-\max_G |G|\theta^* \leq (n-|G_i|)\theta^*. \tag{142}$$

Rearranging, we have

$$n\theta^* \geq 1-(\max_G |G|-|G_i|)\theta^* > 1-(\max_G |G|-|G_i|)/n, \tag{143}$$

where the second inequality follows by the assumption $\theta^* < 1/n$. Substituting this last inequality into the left hand side of (140), we have

$$\theta^*(k-|G_i|+1) \geq \frac{k}{n}\left(1-\frac{|G_i|-1}{k}\right)\left(1-\frac{\max_G |G|-|G_i|}{n}\right) \tag{144}$$

$$\geq \frac{k}{n}\left(1-\frac{|G_i|-1}{k}-\frac{\max_G |G|-|G_i|}{n}\right). \tag{145}$$

The expression at the right hand side is decreasing in $|G_i|$, hence, minimized at $|G_i| = \max_G |G|$. Substituting $|G_i| = \max_G |G|$ above yields

$$\frac{k}{n}\left(1-\frac{\max_G |G|-1}{k}\right) \geq \frac{k}{n}\left(1-\kappa\right). \tag{146}$$

The inequality follows as our instance is in $I(\kappa,\alpha)$, hence

$$\frac{\max_G |G|-1}{k} \leq \kappa. \tag{147}$$

□

C.2.4 Tightness.

Proposition C.8. *For any $\alpha, \kappa \in (0,1)$ and $\epsilon > 0$, there exists an instance in $I(\kappa,\alpha)$ such that the utilization of the group request equilibrium outcome of the Weighted Individual Lotteryis less than $g(\alpha)+\epsilon$.*

Proof of Proposition C.8. Fix $\alpha, \kappa \in (0,1)$ and $\epsilon > 0$. For any instance I, we let $U(I)$ be the utilization of the group request equilibrium outcome under the Weighted Individual Lottery. We will construct a sequence of instances $\{I_\eta\}$ such that for any $\eta \in \mathbb{N}$,

$$I_\eta \in I(\kappa,\alpha) \text{ and } \lim_{\eta\to\infty} U(I_\eta) \to g(\alpha).$$

In I_η, there are $n_\eta = m_\eta s_\eta$ agents divided in m_η groups of size s_η, and $k_\eta = \alpha m_\eta s_\eta$ tickets. We define $\{m_\eta\}, \{s_\eta\}$ to be increasing sequences of natural numbers that satisfy three conditions:

(1) Each instance has an integer number of tickets, that is, $\{m_\eta\}$ must be such that

$$\alpha m_\eta \in \mathbb{N}. \tag{148}$$

(2) Each instance is in $I(\kappa,\alpha)$, i.e.,

$$\frac{k_\eta}{n_\eta} \leq \alpha, \quad \frac{s_\eta-1}{k_\eta} \leq \kappa.$$

The first condition holds immediately as

$$\frac{k_\eta}{n_\eta} = \frac{\alpha m_\eta s_\eta}{m_\eta s_\eta} = \alpha.$$

To ensure the second condition, we will define m_1 to be such that

$$\alpha m_1 \geq \kappa^{-1}. \tag{149}$$

Observe that

$$\frac{s_\eta - 1}{k_\eta} = \frac{s_\eta - 1}{\alpha m_\eta s_\eta} \leq \frac{1}{\alpha m_\eta} \leq \frac{1}{\alpha m_1} \leq \kappa.$$

The second inequality follows as $\{m_\eta\}$ is increasing, and the third by condition (149).

(3) Both sequences grow at a similar rate, more precisely, there exists a positive constant c such that

$$\frac{m_\eta}{s_\eta} \leq c. \tag{150}$$

We will define both sequences explicitly when α is rational. In this case, there exists $p, q \in \mathbb{N}$ such that $\alpha = p/q$. Then, we can let

$$m_\eta = 2\eta q \lceil \kappa^{-1} \rceil, \quad s_\eta = \eta q \lceil \kappa^{-1} \rceil.$$

It's easy to see that conditions (148), (149) and (150) holds, in the third condition $c = 2$. If α is irrational, then we can choose a rational number $\alpha^* \leq \alpha$ that is arbitrarily close to α. We let the number of tickets be $k_\eta = \alpha^* m_\eta s_\eta$ and define m_η in the same way as before but with respect to α^*.

Under the Weighted Individual Lottery, we will draw αm_η agents that get a full allocation. In this context, the utility of each agent is

$$u_i(\pi_i^{SPL}(\mathbf{a})) = 1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right). \tag{151}$$

Therefore, the utilization of this system correspond to

$$U(I_\eta) = \frac{n_\eta}{k_\eta}\left(1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right)\right) = \frac{1}{\alpha}\left(1 - \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right)\right). \tag{152}$$

We claim that

$$\lim_{\eta \to \infty} U(I_\eta) \to g(\alpha).$$

Observe that

$$\begin{aligned}
\liminf_{\eta \to \infty} \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right) &\geq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta}\right)^{\alpha m_\eta} = e^{-\alpha}, \\
\limsup_{\eta \to \infty} \prod_{i=0}^{\alpha m_\eta - 1} \left(1 - \frac{1}{m_\eta - i/s_\eta}\right) &\leq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta - \frac{\alpha m_\eta}{s_\eta}}\right)^{\alpha m_\eta - 1} \\
&\leq \lim_{\eta \to \infty} \left(1 - \frac{1}{m_\eta - \alpha c}\right)^{\alpha m_\eta - 1} = e^{-\alpha}.
\end{aligned}$$

The last inequality follows by condition (150). □