# Parallel Lotteries:
# Insights from Alaskan Hunting Permit Allocation

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**Abstract**

We analyze the parallel lottery which is used to allocate hunting permits in the state of Alaska. Each participant is given tickets to distribute among lotteries for different types of items. Participants who win multiple items receive their favorite, and new winners are drawn from the lotteries with unclaimed items.

When supply is scarce, equilibrium outcomes of parallel lotteries approximate a competitive equilibrium from equal incomes (CEEI), which is Pareto efficient. When supply is moderate, parallel lotteries exhibit two sources of inefficiency. First, some agents may benefit from trading probability shares. Second, outcomes may be "wasteful": agents may receive nothing even if acceptable items remain unallocated. We bound both sources of inefficiency, and show that each is eliminated by giving applicants a suitable number of tickets $k$: trades are never beneficial when $k = 1$, and waste is eliminated as $k \to \infty$.

In addition, we show that the wastefulness of the $k$-ticket parallel lottery has some benefits: agents with strong preferences may prefer parallel lottery outcomes to those of any nonwasteful envy-free mechanism. These agents prefer small values of $k$, while agents with weak preferences prefer large values of $k$. Together, these results suggest that the $k$-ticket parallel lottery performs well under most circumstances, and may be suitable for other settings where items are rationed.

## 1 Introduction

We address the challenge of allocating heterogeneous resources to unit-demand agents without using money. Real-world examples include the allocation of hunting, hiking, camping and fishing permits, affordable housing units, seats at public schools, and scholarships.

We take inspiration from the allocation of hunting permits. In the United States, this allocation is conducted by state agencies. Prospective hunters must apply for a permit, which are limited in supply. In many cases, permitting fees are nominal, and demand exceeds supply. Most states conduct annual lotteries to award permits. Though procedures vary, states generally conduct separate allocations for each species, so that a hunter might apply and win permits for multiple species. However, for each species, hunters are constrained to receive at most one permit.

In Alaska, the allocation for each individual species proceeds as follows. The state offers different types of permit, which specify when and where hunting may occur, as well as the number and gender of animals that may be killed. It sets a quota for each permit type, and publishes this information in an annual "Draw Supplement."[1] Each applicant can submit up to six "applications," and may apply for a given type of permit multiple times to increase their odds of winning (for example, one might apply five times for a popular permit, and once for a second, less popular permit). Each applicant also submits a preference ranking over permits for which they have applied. After the application deadline, applications are drawn randomly for each permit. Applicants who win multiple permits keep only their favorite among these, and new names are drawn for the remainder. This process continues until each permit type has either been fully allocated or offered to all applicants.[2]

This inspires several natural questions. How do outcomes of this lottery compare to those that would result from using other procedures? Are these outcomes efficient, or at least approximately so? And what would be the effect of increasing or decreasing the maximum number of applications?

We address these questions using a model with a continuum of agents and $n$ types of items. In Section 3 we define the *k-ticket parallel lottery*, in which agents may allocate $k \in \mathbb{N}$ tickets across lotteries. Given agents' actions, each item has a resulting level of competition, summarized by the probability that a ticket entered into the corresponding lottery will be drawn. Agents take these win probabilities into account when deciding how to use their tickets.

Our first result establishes that when there are many more agents than items, equilibrium outcomes of the $k$-ticket parallel lottery are close to outcomes of a competitive equilibrium with equal incomes (CEEI): each agent is nearly indifferent between these mechanisms. Thus, when demand is high, the choice of $k$ is not very important, and parallel lotteries can be thought of as a virtual currency system. In particular, their equilibrium outcomes are approximately Pareto

[1] The most recent supplement is available at `https://www.adfg.alaska.gov/index.cfm?adfg=huntlicense.drawsupplements`, and additional information on the draw can be found at `https://www.adfg.alaska.gov/index.cfm?adfg=huntlicense.lottery`.

[2] An alternative description of the process is as follows. A hunter who submits $t_i$ applications to permit type $i$ is given a random priority for $i$ that is the first-order statistic of $t_i$ iid uniform draws from $[0, 1]$. Using these priorities and the preference rankings submitted by hunters, the state runs the permit-proposing Deferred Acceptance algorithm.

efficient.

In general, parallel lotteries exhibit two forms of inefficiency. First, some agents may benefit from trading probability shares. Second, outcomes may be "wasteful": agents may receive nothing even if acceptable items remain unallocated. Our second contribution is to bound both sources of inefficiency. In particular, Theorem 3 states that for any equilibrium allocation, no ex ante exchange of probability shares can increase the welfare of every agent to more than $\frac{1}{1-(1-1/k)^k}$ times their welfare under the original allocation. Theorem 5 states that no reallocation of wasted items can increase the welfare of any agent to more than $(1+\frac{1}{k})$ times their welfare under the original allocation. In particular, these two theorems imply that when $k=1$, no beneficial trades are possible, and as $k \to \infty$, waste is eliminated. In addition, we prove that the $k$-ticket parallel lottery is $\frac{1}{1-(1-\frac{1}{k+1})^k}$-Pareto efficient (Theorem 2).

Our third contribution is to demonstrate that the wastefulness of parallel lotteries may actually improve sorting: agents with strong preferences for popular items may prefer the $k$-ticket parallel lottery to any nonwasteful mechanism. We illustrate this by considering the scenario in which there are two types of items and one item is universally preferred, while the other is abundant. In this scenario, CEEI is equivalent to random matching. (This occurs because the less demanded item is available at no cost, so all agents spend their entire budget on the highly demanded item.) By contrast, in a $k$-ticket parallel lottery, securing the second item comes at the cost of spending a ticket, which only agents with a weak preference for the first item are willing to pay. Proposition 5 establishes that in this setting, smaller choices of $k$ are better for agents who strongly prefer the more popular item. In addition, we demonstrate that no other mechanism achieves greater efficiency with the same amount of waste: Proposition 4 states that in the two-item setting, allocating items with a parallel lottery is equivalent to discarding the items wasted by the parallel lottery and then running CEEI.

We provide advice for choosing $k$ and practical recommendations for implementing the $k$-ticket parallel lottery in Sections 6 and 7. In Section 8, we apply our decision criteria to provide provisional recommendations for several real-world settings.

Taken together, our results suggest that parallel lotteries produce good outcomes across a range of market conditions. When demand far outpaces supply, they approximate a competitive equilibrium from equal incomes. When demand and supply are more balanced, they remain approximately efficient and may allow agents to signal their preference intensities more effectively than CEEI.

## 2 Related Work

Reeling and Verdier (2018) also study the allocation of hunting permits. They use bear hunting permit data from the Michigan Department of Natural Resources to calibrate a structural model, and then use this model to compare alternative allocation policies. They highlight that the current dynamic mechanism allows hunters to match efficiently across time.

Hylland and Zeckhauser (1979) introduce Competitive Equilibrium from Equal Incomes (CEEI), in which agents use virtual currency to buy probability shares of items. Equilibria of CEEI are ex-ante Pareto efficient and envy-free, and Ashlagi and Shi (2015) show that this is the only mechanism with these properties. Despite its virtues, CEEI is rarely used in practice due to the challenges of soliciting preference intensities from agents.

Several papers have argued that other mechanisms may approximate CEEI outcomes. Abdulkadiroğlu et al. (2011) and Miralles (2009) note advantages of the "Boston" mechanism, in which items are preferentially awarded to agents who rank them highly. Abdulkadiroğlu et al. (2015) introduce Choice-Augmented Deferred Acceptance (CADA), which allows agents to improve their priority at one targeted item. If items have no inherent preferences over agents, the Boston mechanism is simply a sequence of 1-ticket parallel lotteries, and CADA is a 1-ticket parallel lottery followed by random serial dictatorship. Due to the close relationship between these mechanisms and the 1-ticket parallel lottery, our claim that the one-ticket lottery is trade efficient (Corollary 1) is analogous to the observation that schools that fill in the first round of Boston are efficiently allocated (Miralles, 2009), as well as an analogous claim for CADA (Abdulkadiroğlu et al., 2015).

Immorlica et al. (2017) introduce the "raffle", which is the natural extension of the $k$-ticket parallel lottery with $k = \infty$. They show that for any equilibrium of this mechanism, it is impossible to simultaneously increase all participants' welfare to $\frac{e}{e-1}$ times their original welfare, and argue that this mechanism is a practical alternative to CEEI. However, it is hard to imagine deploying a mechanism that requires applicants to allocate an infinitely divisible mass of tickets. As such, our work can be viewed as a study of more practical variants of the raffle.

One key difference between a $k$-ticket parallel lottery and the $\infty$-ticket raffle is that the former might fail to allocate items, even if there are unmatched agents who consider them acceptable. In fact, this wastefulness also differentiates the parallel lottery from CEEI, the Boston mechanism, CADA, and Random Serial Dictatorship. Although wastefulness seems to be a shortcoming, Section

6 shows that it also has benefits: parallel lotteries may achieve more effective sorting than these other mechanisms, benefiting agents with strong preferences. The setting in Section 6 is very similar to that considered by Cavallo (2014). He seeks to maximize utilitarian welfare, and shows that in some cases, random allocation is optimal, but in others, wasteful mechanisms offer improvements.

At a technical level, our paper leverages several results from other papers. Lemma 1, which gives an agent's best response, builds upon the work of Chade and Smith (2006). Lemma 2, which states that every strategy profile induces a unique consistent outcome, builds upon the uniqueness of a stable outcome established by Azevedo and Leshno (2016). The proof that 1-ticket lotteries have a unique equilibrium is similar to a result in Gale (1976). The proofs of Theorems 2, 3, and 4 are inspired by the proof of approximate efficiency in Immorlica et al. (2017).

# 3 Model

There are $n$ types of items with quantities $\mu = (\mu_1, \mu_2, ..., \mu_n) \in \mathbb{R}^n_+$. There is mass $M$ of agents. Each agent is identified by a type $v = (v_1, v_2, ...v_n) \in \mathbb{R}^n$, where $v_i$ denotes the agent's value for item $i$. Types are distributed according to a probability measure $\eta$ that is absolutely continuous with respect to the Lebesgue measure. Agents also have an outside option with a value normalized to zero. Without loss of generality, we consider only agent types with positive value for at least one item and items which a positive measure of agents prefer to their outside option.

In Section 3.1, we define the optimization problem facing a single agent. Section 3.2 discusses how agent strategies determine aggregate outcomes, and defines our equilibrium concept. Section 3.3 introduces several efficiency metrics.

## 3.1 Single-Agent Decision Problem

The agent's interaction with the $k$-ticket parallel lottery is summarized by a strategy $s = (t, \succ)$ consisting of a ticket allocation vector $t = (t_1, t_2, ..., t_n)$ and a preference ranking $\succ$ over the set of items. For $i, j \in [n]$, $t_i$ denotes the number of tickets entered in lottery $i$ and $j \succ i$ indicates that the agent prefers $j$ to $i$ in her preference ranking. Let $S_k$ denote the set of all possible strategies with at most $k$ tickets. A *strategy profile* $\Phi : \mathbb{R}^n \to S_k$ associates a $k$-ticket strategy with every agent type.

A strategy profile $\Phi$ induces a vector $p = (p_1, p_2, ..., p_n)$ of win probabilities, where $p_i$ denotes the probability that an individual ticket entered in lottery $i$ will be drawn during the lottery resolution process. Agents take $p$ as given when considering which strategy to adopt. We address the calculation of $p$ in Section 3.2.

Given strategy $s$ and win probabilities $p$, the vector $\chi(s,p) = \{\chi_i(s,p)\}_{i=1}^n$ summarizes the probability that an agent who adopts strategy $s$ wins each item. We define

$$\chi_i(s,p) := (1 - (1-p_i)^{t_i}) \prod_{j \succ i} (1-p_j)^{t_j}, \tag{1}$$

meaning that an agent wins item $i$ so long as at least one the tickets entered in lottery $i$ is drawn and the agent does not win any item $j$ for which $j \succ i$.

For fixed $p$, the expected welfare of an agent with type $v$ under strategy $s$ is

$$w(v,s,p) := v \cdot \chi(s,p). \tag{2}$$

At any equilibrium strategy profile, each agent maximizes his or her expected welfare.

**Definition 1.** *Let $p = (p_1, p_2, ..., p_n)$ be a fixed vector of win probabilities for each lottery. A strategy profile $\Phi$ is* ***optimal*** *given $p$ if*

$$\forall v, \Phi(v) \in \arg\max_{s \in S_k} w(v,s,p). \tag{3}$$

If agents have a single ticket, the optimal strategy is clear: spend it on the item $i^*$ that maximizes $p_i v_i$. With multiple tickets, one might conjecture that the optimal strategy spends them all on $i^*$. However, this is not necessarily the case. Because an agent might have multiple tickets drawn, but can keep only one item, tickets have diminishing marginal returns. This gives agents an incentive to spend tickets on items which are both more desirable and more competitive than $i^*$.

Despite this complexity, an agent's optimal strategy can be computed by a simple greedy algorithm. The algorithm starts by spends its first ticket on $i^*$. It then allocates each subsequent ticket to maximize the marginal gain in utility from that ticket. As a consequence, later tickets are spent on increasingly competitive items.

Below, we write $s + \{i\}$ to denote the strategy $s$ with an additional ticket placed in the lottery for item $i$.

**Lemma 1.** *For any $p$ and any $v$, the following greedy algorithm computes an optimal strategy:*

1. *Set $s := (\vec{0}, \succ)$, where $\succ$ is ordered according to the true preferences of $v$.*

2. *For $j = 1, 2, .., k$: select $i_j \in \arg\max_{i \in [n]}(w(v, s + \{i\}, p) - w(v, s, p))$. If the $\arg\max$ is not unique, select $i_j$ to be the agent's most preferred item in this set. Set $s := s + \{i_j\}$.*

*Proof Sketch.* The problem of computing an optimal $k$-ticket strategy can be transformed into a downward-recursive portfolio choice problem over stochastic options. Lemma 1 then follows from the optimality of the Marginal Improvement Algorithm introduced by Chade and Smith (2006). The reduction to downward recursive portfolio choice is elaborated in Appendix A.1. □

Multiple choices of $i$ may maximize $w(v, s + \{i\}, p)$ for two reasons. First, different strategies may lead to the same probabilistic allocation if the agent can secure his or her most preferred item with a single ticket. In this case, the algorithm above ensures that agents continue to enter tickets into the lottery for their most preferred item.[3] Second, an agent may be exactly indifferent between two strategies that lead to different allocations. Because $\eta$ is absolutely continuous, for any $p$ this occurs only for a set of agents with measure zero. Thus, the algorithm above uniquely determines allocation probabilities for almost every agent type.

### 3.2 $k$-Ticket Equilibria

Given a strategy profile, any win probability vector $p$ must be consistent with the available quantities $\mu$. That is, for $i \in [n]$, either the entire mass $\mu_i$ should be allocated or every ticket entered into lottery $i$ should be drawn.

**Definition 2.** *Fix a strategy profile $\Phi$. A vector $p$ of win probabilities is **consistent** with $\Phi$ if for all $i \in [n]$,*

$$M \sum_{s \in S_k} \chi_i(s, p)\, \eta(\{v : \Phi(v) = s\}) \leq \mu_i, \tag{4}$$

*with equality if $p_i < 1$.*

**Lemma 2.** *For each strategy profile, there is a unique consistent win probability vector.*

[3] This choice is motivated by the fact that this strategy is uniquely optimal under a small perturbation of the vector $p$.

*Proof Sketch.* For a fixed strategy profile $\Phi$, the $k$-ticket parallel lottery is equivalent to a deferred acceptance procedure in which agents' priority for each item $i$ increases stochastically with the number of tickets placed in the lottery for item $i$. It then follows from the results of Azevedo and Leshno (2016) that every strategy profile $\Phi$ admits at least one vector $p$ of win probabilities, that the set of win probability vectors forms a lattice, and that every consistent vector $p$ matches the same set of agents. Because the measure of matched agents is strictly increasing in $p$, the lattice property implies that the consistent $p$ is unique. The full proof is in Appendix A.2. □

**Definition 3.** *A **k-ticket equilibrium** is a pair* $(\Phi, p)$ *consisting of a strategy profile and a vector of win probabilities such that* $\Phi$ *is optimal given* $p$ *and* $p$ *is consistent with* $\Phi$*.*

**Proposition 1** ($k$-Ticket Equilibria Exist)**.** *Every $k$-ticket lottery admits an equilibrium.*

*Proof Sketch.* We describe a continuous function $f$ that maps the space of win probability vectors to itself such that $f(p) = p$ if and only if $p$ corresponds to a $k$-ticket equilibrium $(\Phi, p)$. Applying Brouwer's fixed point theorem to $f$ demonstrates that an equilibrium exists. The full proof is in Appendix A.2. □

In fact, the 1-ticket parallel lottery has a unique vector $p$ of win probabilities at equilibrium. When $k \geq 2$, however, parallel lotteries may have multiple equilibria with distinct win probability vectors. We prove the uniqueness of 1-ticket equilibria and provide an example illustrating the non-uniqueness of 2-ticket equilibria in Appendices A.3 and A.4.

### 3.3 Envy-Free and Pareto-Efficient Allocations

A (probabilistic) allocation is a function $x : \mathbb{R}^n \to [0,1]^n$ that assigns each agent a chance to receive each item. Each agent type $v$ is allocated $x(v) = (x_1(v), x_2(v), ..., x_n(v))$, where $x_i(v)$ denotes the probability of receiving item $i$. Allocations satisfy the following feasibility constraints:

1. Allocations respect the quantity of each item:

$$\forall i, M \int x_i(v) d\eta \leq \mu_i. \tag{5}$$

2. Each agent receives at most one item:

$$\forall v, \sum_{i=1}^{n} x_i(v) \leq 1. \tag{6}$$

**Definition 4.** *When the inequality in* (5) *is tight, we say that item $i$ is* **fully allocated** *under the allocation $x$. Otherwise, we say that item $i$ is* **underdemanded***.*

A $k$-ticket equilibrium $(\Phi, p)$ corresponds to the allocation $x$ such that $x(v) = \chi(\Phi(v), p)$.

**Definition 5.** *An allocation is* **envy-free** *if every agent weakly prefers her allocation to that of every other agent. That is, for every agent type $v$, we have*

$$x(v) \cdot v = \max_{u \in \mathbb{R}^n} x(u) \cdot v. \tag{7}$$

The property of envy-freeness is readily interpreted as a fairness constraint as it ensures that no agent envies the allocation of another. However, it also arises naturally if the mechanism designer cannot observe agent types, and must offer the same set of options to every agent. From this perspective, envy-freeness may be thought of as a feasibility constraint that enforces anonymity. In our continuum model, allocations produced by the $k$-ticket lottery are envy-free because every agent faces the same win probability vector $p$.

**Definition 6.** *Allocation $y$* ***Pareto dominates*** *allocation $x$ if $y(v) \cdot v \geq x(v) \cdot v$ for every $v$, with strict inequality for some set of agents with positive $\eta$-measure. An allocation is* ***Pareto-efficient*** *if no feasible allocation Pareto dominates it.*

## 4 The High Demand Setting

We start by considering the case where the number of agents is much larger than the number of items. In this case, the win probability for each item is small. As a result, an agent's chance of receiving an item is roughly proportional to the number of tickets spent on that item. In other words, tickets serve as virtual currency, and agents spend their tickets on the single item that maximizes "bang for buck." The resulting outcome resembles a competitive equilibrium from equal incomes. We establish this formally by showing that as $M$ grows, each agent becomes indifferent between the $k$-ticket parallel lottery and CEEI.

**Theorem 1.** *Fix $k$, $\mu$, and $\eta$. Let $X_M$ denote the set of allocations corresponding to $k$-ticket equilibria with the given parameters and agent mass $M$, and let $ce_M$ denote the unique allocation under CEEI.*[4] *For almost every agent $v$, we have*

$$\lim_{M\to\infty} \sup_{x\in X_M} \frac{x(v)\cdot v}{ce_M(v)\cdot v} = \lim_{M\to\infty} \inf_{x\in X_M} \frac{x(v)\cdot v}{ce_M(v)\cdot v} = 1. \tag{8}$$

*Proof sketch.* According to the greedy algorithm for optimal strategy choice (Lemma 1), agents place tickets in more than one lottery if the difference in marginal benefit between placing their first and last ticket in the same lottery is sufficient to justify the change. If the odds of winning each lottery are small, the marginal benefit of placing each subsequent ticket in the same lottery decreases very slowly. As a result, most agents opt to place all their tickets in a single lottery and the resulting outcome is similar to the unique 1-ticket equilibrium. Finally, when all items are fully allocated, the 1-ticket equilibrium allocation is equivalent to CEEI. The full proof of Theorem 1 is in Appendix B. □

## 5 Approximate Efficiency

Section 4 establishes that when there are far more agents than items, any $k$-ticket equilibrium outcome approximates CEEI, which is Pareto efficient. In this section, we make no assumptions about the number of agents and items. Although $k$-ticket equilibria are not Pareto efficient, we prove that they are approximately so. We adopt the notion of approximate Pareto efficiency proposed by Immorlica et al. (2017), which says that an allocation is $\alpha$-Pareto efficient if it is impossible to simultaneously improve each agent's welfare by a multiplicative factor of $\alpha$.

**Definition 7.** *Allocation $x$ is $\alpha$-Pareto efficient if for every feasible allocation $y$*

$$\eta\left(\left\{v : \frac{y(v)\cdot v}{x(v)\cdot v} \le \alpha.\right\}\right) > 0. \tag{9}$$

Immorlica et al. (2017) show that equilibria of the $\infty$-ticket lottery are $e/(e-1) \approx 1.58$-Pareto efficient. Our next result establishes approximate Pareto efficiency for any $k$-ticket equilibrium. It implies that equilibria of the the 1-ticket lottery are 2-Pareto efficient. The approximation factor

[4] In general, CEEI may have multiple equilibria. However, if there is an equilibrium in which all items are fully allocated, then this equilibrium is unique. Therefore, for all sufficiently large $M$, $ce_M$ is well defined.

improves as $k$ increases, and converges to $e/(e-1)$ as $k \to \infty$.

**Theorem 2.** *Any equilibrium of the $k$-ticket parallel lottery is $\frac{1}{1-(1-\frac{1}{k+1})^k}$-Pareto efficient.*

The proof of Theorem 2 is located in Appendix C.4. Although this result bounds the inefficiency of any $k$-ticket equilibrium, it does not explain *why* these equilibria may be inefficient in the first place. Our next step is to identify two sources of inefficiency that may arise. First, two or more agents may be able to profitably exchange probability shares. Second, there might be unallocated items which can be assigned to benefit some agents. We formalize these concepts below.

**Definition 8.** *A* **trade reallocation** *of $x$ is a feasible allocation $y$ such that $\int x(v)\, d\eta = \int y(v)\, d\eta$. Allocation $x$ is $\alpha$-***trade efficient** *if for any $y$ that is a trade reallocation of $x$,* (9) *holds. An allocation is* **trade efficient** *if it is* 1*-trade efficient.*

**Definition 9.** *A* **waste reallocation** *of $x$ is a feasible allocation $y$ such that for every $v$ and every $i$ that is fully allocated under $x$, $y_i(v) \leq x_i(v)$. Allocation $x$ is $\epsilon$-***wasteful** *if for any $y$ that is a waste reallocation of $x$, and any $v$, $\frac{v \cdot y(v)}{v \cdot x(v)} \leq 1 + \epsilon$. An allocation is* **nonwasteful** *if it is* 0*-wasteful.*

Definition 9 implies that an allocation is wasteful if there is some item that is not fully allocated and some agent with a positive probability of an outcome that is worse than receiving this item. It quantifies waste not by the number of unallocated items, but rather by the value that an agent could obtain from them.

If agents outnumber items and always prefer something to nothing, nonwastefulness and trade-efficiency are equivalent to the familiar property of Pareto efficiency.

**Proposition 2.** *If $M \geq \sum_i \mu_i$ and all agents have positive values for all items, an allocation $x$ is Pareto efficient if and only if it is nonwasteful and trade efficient.*

*Proof.* Pareto efficiency trivially implies trade efficiency and nonwastefulness. In this setting, any nonwasteful mechanism allocates all items. When all items are allocated, trade efficiency implies Pareto efficiency. □

In the remainder of this section, we establish that $k$-ticket equilibria are approximately trade efficient and nonwasteful. When $k = 1$, the equilibrium allocation is perfectly trade efficient, and the allocation becomes nonwasteful as $k \to \infty$. For intermediate $k$, both sources of inefficiency are present, but modest in magnitude. Our findings are summarized in Figure 1.

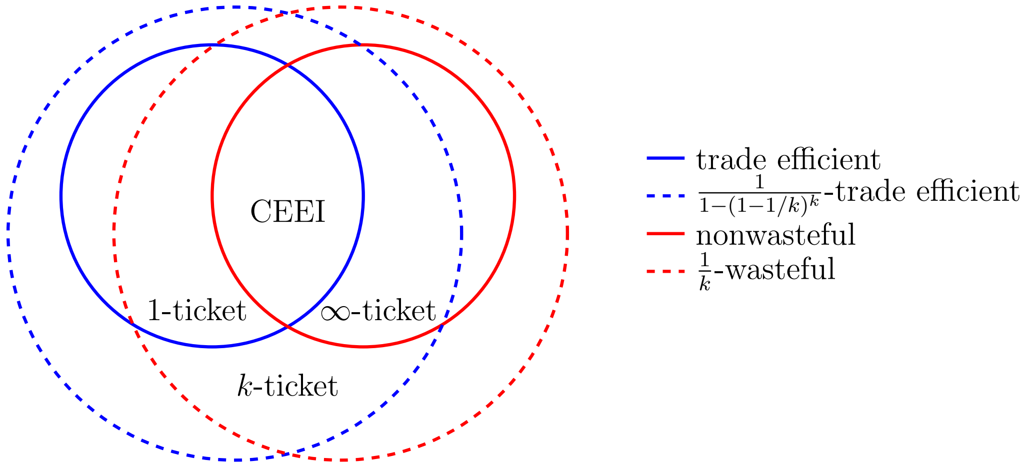



Figure 1: CEEI is both trade efficient and nonwasteful. Although parallel lotteries do not generally satisfy either of these conditions, they are approximately trade efficient and approximately nonwasteful. Furthermore, trade efficiency is attained for $k = 1$ and nonwastefulness as $k \to \infty$.

### 5.1 Approximate Trade Efficiency

The $k$-ticket parallel lottery is typically not trade efficient. This is because agents may have multiple items selected, but can keep only one. As a result, each additional ticket delivers diminishing marginal value. This distortion incentivizes agents to spend later tickets on increasingly competitive items, creating the possibility of mutually beneficial trades.[5]

To illustrate, consider a setting with two items and an equilibrium of the 2-ticket parallel lottery in which $p_1 = 1/4, p_2 = 2/3$. Note that the fractional allocations $(1/2, 0)$ and $(0, 1)$ are not feasible. However, the average of these allocations, $(1/4, 1/2)$, is obtained by spending one ticket on each item and ranking item one first. Two agents who both adopt this strategy but have different relative values for items 1 and 2 may wish to trade so that the agent with a stronger preference for item 1 receives allocation $(1/2, 0)$ and the agent with weaker preference for item 1 receives allocation $(0, 1)$.[6]

In fact, the opportunity for mutually beneficial exchange of probability shares occurs whenever non-identical agents both place tickets in lotteries for items $i$ and $j$ and are not assured of being allocated (see Appendix C.2 for details). In this section, we show that although mutually beneficial

[5] That is, there are mutually beneficial trades of probability shares ex ante. The $k$-ticket parallel lottery does satisfy the weaker notion of *ex post* trade efficiency: after the allocation occurs, no agents can exchange items to their mutual benefit. For details, see Appendix C.1.

[6] To make this concrete, suppose that Agent 1 has value $v^1 = (8/3, 1)$ and Agent 2 has value $v^2 = (5/3, 1)$. In a 2-ticket equilibrium with $p_1 = 1/4, p_2 = 2/3$, both agents find it optimal to spend one ticket on each item and receive allocation $(1/4, 1/2)$. This results in utilities of $7/6$ and $11/12$, respectively. Reallocating these probability shares to give Agent 1 allocation $(1/2, 0)$ and Agent 2 allocation $(0, 1)$, increases their utilities to $8/6$ and 1, respectively.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 10 | 15 | 20 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| Lower Bound | 1 | 1.207 | 1.189 | 1.166 | 1.155 | 1.150 | 1.142 | 1.136 | 1.132 | 1.125 |
| Upper Bound | 1 | 1.207 | 1.421 | 1.463 | 1.487 | 1.504 | 1.535 | 1.551 | 1.559 | 1.577 |

Table 1: Bounds on approximate trade efficiency of $k$-ticket parallel lotteries. The upper bounds are established in Theorems 3 and 4. The lower bounds are derived from an exhaustive search in a setting with two types of items, in the limit as the win probability of the more competitive item tends to zero. For $k = 1, 2$, the upper and lower bounds coincide. For $k > 2$, we conjecture that the lower bounds are tight.

trades frequently exist, the potential benefit from these trades is modest.

**Theorem 3.** *Any equilibrium allocation of the $k$-ticket parallel lottery is $\frac{1}{1-(1-1/k)^k}$-trade efficient.*

We prove Theorem 3 in Appendix C.3.1. For $k = 1$ we obtain the following corollary.

**Corollary 1.** *The 1-ticket equilibrium allocation is trade-efficient.*

This result has practical importance for a mechanism designer. Although the 1-ticket parallel lottery is only 2-Pareto efficient, it is perfectly trade efficient. Therefore, in settings where waste is not a concern, the 1-ticket lottery may be good choice.

Although the bound in Theorem 3 is not tight, it approaches the bound of Immorlica et al. (2017) for the $\infty$-ticket lottery as $k \to \infty$. Table 1 provides lower bounds on the trade efficiency of the $k$-ticket lottery. To generate these bounds, we consider the set of markets with two items exchanged by two agent types. We narrow the parameter space by assuming that both agent types trade until they have a chance to win only one of the two goods, that agents who trade for the less congested good are guaranteed to receive it after trading, and that $p_1 \to 0$. With these assumptions, searching over all possible strategies and values $p_2 \in [0, 1]$ yields the lower bounds in Table 1.

We conjecture that the lower bounds in Table 1 are tight. As evidence for this conjecture, we prove that our lower bound for the $k = 2$ case is tight.

**Theorem 4.** *Any equilibrium of the 2-ticket parallel lottery is $\frac{1+\sqrt{2}}{2}$-trade efficient. For any $\epsilon > 0$, there exists an equilibrium of the 2-ticket parallel lottery that is not $(\frac{1+\sqrt{2}}{2} - \epsilon)$-trade efficient.*

The proof of Theorem 4 is located in Appendix C.3.1.

## 5.2 Approximate Nonwastefulness

We now show that no waste reallocation of a $k$-ticket equilibrium can increase any agent's welfare by more than $1/k$.

**Theorem 5.** *Any equilibrium allocation of the $k$-ticket parallel lottery is $1/k$-wasteful.*

*Proof.* Let $x$ be an allocation corresponding to a $k$-ticket equilibrium and $A_x$ be the set of fully allocated items under $x$. For a given $v$, the best possible waste reallocation ensures that $v$ receives $i^* \in \arg\max_{i \in [n] \setminus A_x} v_i$ if she fails to win anything else, and thus

$$y(v) \cdot v = x(v) \cdot v + (1 - \sum_i x_i(v)) v_{i^*}. \tag{10}$$

Consider the process of optimal strategy selection outlined in Lemma 1. At each stage, the agent chooses to place her $j^{th}$ ticket in the lottery corresponding to the item which maximizes her marginal increase in welfare. At each stage, the agent declines to place a ticket in the lottery for item $i^*$, so her marginal increase in welfare must be at least $(1 - \sum_i x_i(v)) v_{i^*}$ at each step. It follows that her total welfare

$$x(v) \cdot v \geq k \cdot (1 - \sum_i x_i(v)) v_{i^*}. \tag{11}$$

The theorem follows from substituting (11) into (10). □

Theorem 5 is in fact tight.[7] It implies that the mechanism designer can reduce waste to an arbitrarily small fraction of welfare by choosing a large value of $k$. The next section explains why this may not lead to desirable outcomes.

## 6 How To Choose $k$

When demand far outpaces supply, Section 4 shows that any $k$-ticket parallel lottery produces similar outcomes to CEEI. In such cases, the parallel lottery is approximately trade efficient and nonwasteful. When supply and demand are more balanced, Section 5 provides worst-case bounds on each of these sources of inefficiency. However, these worst-case results do not directly address the effect of changing $k$ while the setting (specified by the supply $\mu$ and type measure $\eta$) remains fixed. This section addresses that question.

[7] To see this, consider a setting with two types of items, and let $x$ be the allocation corresponding to a $k$-ticket equilibrium with $p = (\epsilon, 1)$. Consider agent type $v = (1/\epsilon, 1 - \epsilon)$, for some small $\epsilon > 0$. By Lemma 1, the optimal strategy is to place all $k$ tickets in the lottery for item 1. As $\epsilon \to 0$, $x(v) \cdot v \to k$. If $y$ gives the agent item 2 whenever she fails to receive item 1, then $y(v) \cdot v \to k + 1$ as $\epsilon \to 0$.

To obtain clean insights, we consider a simple scenario with two types of item. Without loss of generality, we normalize the agent mass $M$ to 1, and assume that all agents have a positive value for both items. This implies that each agent's allocation depends only on the ratio $v := \frac{v_2}{v_1}$, and the measure of types $\eta$ can be summarized by the CDF of $v$, which we denote by $F$. In order to make the setting nontrivial we assume that the number of agents who prefer item 1 exceeds its supply ($F(1) > \mu_1$). We refer to this as the *two-item setting.* We focus on this setting because agent types can be captured by a single number, making it possible to clearly identify agents with the "strongest" preference for item 1 (smallest values of $v$), and to plot welfare by type. However, we believe that many of our qualitative conclusions apply to markets with more item types.

## 6.1 The Case for Waste

Suppose that most agents prefer item 1, with some strongly preferring it to item 2 ($v \ll 1$), while others are nearly indifferent ($v \approx 1$). It seems natural to try to distribute item 1 among the former group, and item 2 among the latter. The following result says that when there is an abundance of item 2, this conflicts with the natural objective of eliminating waste.

**Proposition 3.** *In the two-item setting, if there are more items than agents ($\mu_1 + \mu_2 > 1$), then in every nonwasteful envy-free allocation, all agents receive an item, and item* 1 *is allocated randomly among agents who prefer it to item* 2*.*

*Proof.* Because $\mu_1 + \mu_2 > 1$, there will be excess of either item 1 or item 2. Because $F(1) > \mu_1$, excess of item 1 implies a wasteful allocation. Therefore, there must be excess of item 2, and every agent who prefers item 2 must receive it with certainty. Furthermore, because all agents have positive values for both items, we must have $x_1(v) + x_2(v) = 1$ for all $v$. Envy-freeness thus implies $x_1(v) = x_1(v')$ for every pair of agent types $v, v' < 1$. □

Proposition 3 implies that when there is an abundance of item 2, CEEI, RSD, and the $\infty$-ticket Lottery are all equivalent. Under any of these mechanisms, all agents who prefer item 1 to item 2 have identical ex ante allocations. In contrast, $p_1 < 1$ at equilibrium in the 1-ticket parallel lottery. Here agents who prefer item 1 face a choice: bid on their preferred item, and risk receiving nothing, or bid on item 2, and receive it with certainty. Agents with a strong preference for item 1 will choose the former option, while those with a weak preference will choose the latter. A similar effect occurs for larger values of $k$, though to a lesser extent.

## 6.2 The Case for the $k$-Ticket Parallel Lottery

Proposition 3 implies that waste may be necessary in order to ensure that item 1 is allocated to those who strongly prefer it, and the $k$-ticket lottery may accomplish this goal. However, might there be a better choice of wasteful mechanism than the $k$-ticket lottery? In particular, is it possible to design a mechanism with comparable levels of waste that all agents prefer? Proposition 4 states that in the two-item setting with an abundance of item 2, the answer is no: the $k$-ticket parallel lottery is equivalent to running CEEI after discarding some items, and is therefore trade efficient.

**Proposition 4.** *In the two-item setting, if $\mu_1 + \mu_2 \geq 1$, then each $k$-ticket equilibrium allocation $x$ corresponds to a CEEI with quantity vector $\tilde{\mu} := \int x(v)\, dF(v)$.*

The proof of Proposition 4 is in Appendix D.1. Proposition 4 implies that a version of CEEI with preemptive disposal could replicate outcomes from the $k$-ticket parallel lottery. However, the difficulty of computing $\tilde{\mu}$ in advance makes this an impractical alternative.

## 6.3 Effect of Changing $k$

The preceding subsections argued that waste may be necessary to achieve good sorting, and that $k$-ticket lotteries are natural wasteful mechanisms to consider. We next compare $k$-ticket parallel lottery outcomes, and ask which agents prefer smaller $k$.

One might think that the agents who risk getting nothing in the parallel lottery would prefer a nonwasteful mechanism. However, in many cases the *opposite* is true. Agents who choose not to bid on underdemanded items have relatively little value for them. Moreover, low win probabilities for popular items deter agents with weak preferences for these items from bidding on them, to the benefit of those with stronger preferences. This deterrent effect can happen in any $k$-ticket parallel lottery, but is especially dramatic if $k$ is small. The following result formalizes that in the two-item setting, agents with strong preferences for popular items are most likely to benefit from smaller $k$.

**Proposition 5** (Agents with Strong Preferences Prefer Smaller $k$)**.** *In the two-item setting with $\mu_1 + \mu_2 \geq 1$, let $x^{k_1}$ and $x^{k_2}$ denote allocations corresponding to $k_1$-ticket and $k_2$-ticket equilibria, with $k_1 < k_2$. If an agent with $v = \overline{v}$ prefers $x^{k_1}$ to $x^{k_2}$, then so do all agents with $v \leq \overline{v}$.*

The proof of Proposition 5 is in Appendix D.2. This result suggests that when choosing $k$, there is often a tradeoff: smaller values of $k$ favor agents with a strong preference for item 1, at the expense of those with weaker preferences. Figure 2a illustrates this tradeoff in a market where $v$ is uniformly distributed on the unit interval.

Crucially, although Proposition 5 implies that the agents who prefer smaller $k$ are those with the strongest preferences for item 1, it does not guarantee that any such agents exist. If most agents strongly prefer item 1, then a 1-ticket parallel lottery does not significantly reduce competition for item 1, and merely results in item 2 being wasted. In such cases, increasing $k$ offers unambiguous improvements: it allows some agents to spend a ticket securing item 2, improving outcomes both for them and for agents who spend all of their tickets competing for item 1. Once enough agents choose to spend a ticket on item 2, the tradeoff discussed in the preceding paragraph emerges: further increases in $k$ hurt those with the strongest preferences for item 1, while helping others. This scenario is illustrated in Figure 2b.

The preceding examples illustrate two qualitative regimes:

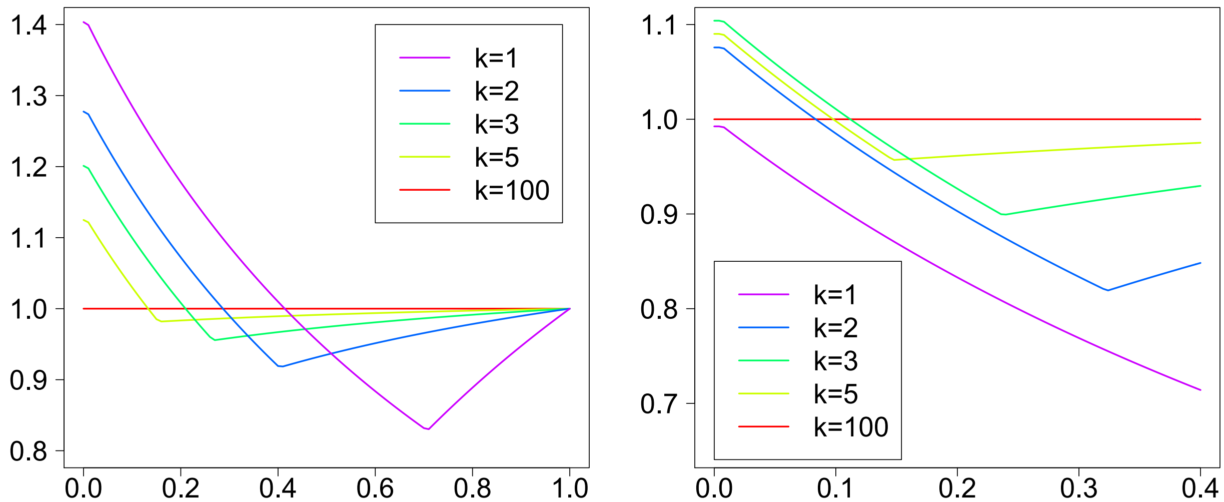
- **Tradeoff Regime.** Many agents are willing to bid on an underdemanded item at equilibrium. Increasing $k$ benefits these agents by allowing them them to compete for overdemanded items. This increased competition hurts agents who strongly prefer the most competitive items.

- **Waste Trap Regime.** Few agents are willing to bid on underdemanded items, most of which go to waste. Increasing $k$ will benefit most or all agents.

These observations offer guidance to a designer choosing $k$. Section 8 describes several applications in which a designer may wish to favor agents with unusually strong preferences. In these cases, the designer should select the smallest value of $k$ that avoids the waste trap, often $k = 1$. Meanwhile, the designer can favor agents with weak preferences by choosing a large $k$. We suspect that the Alaskan hunting permit lottery, which uses $k = 6$, results in similar outcomes to larger values of $k$ and the $\infty$-ticket lottery.

### 6.4 Markets with Moderate Supply

Section 4 considers the case in which the supply of every item is a vanishing fraction of demand, while subsections 6.1-6.3 consider the case in which at least one item is abundant. However, many real-world markets fall into an intermediate regime where supply is sufficient for many but not all agents to receive an item.




(a) $v \sim U[0, 1]$ $\mu = (.5, .5)$

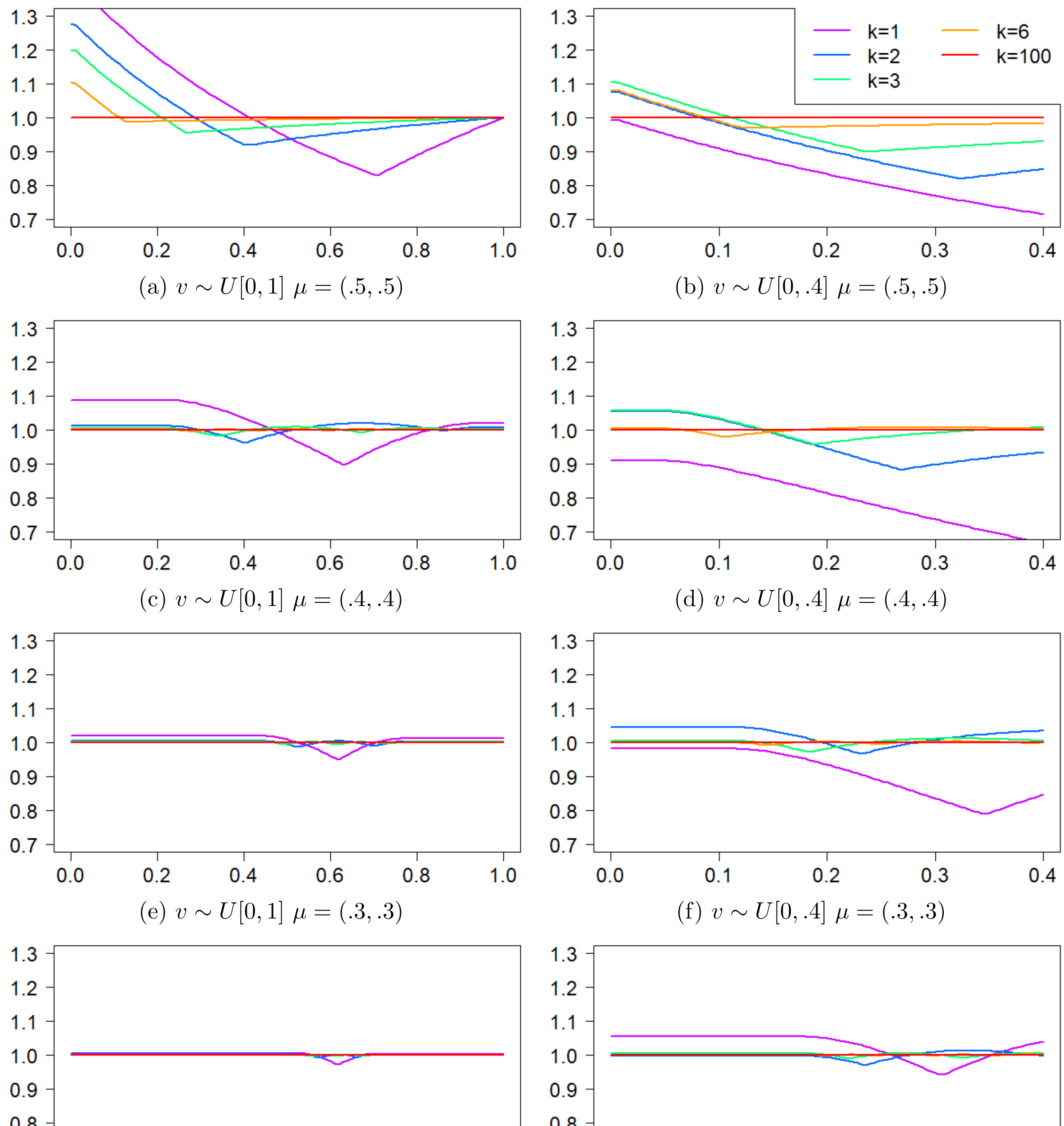
(b) $v \sim U[0, .4]$ $\mu = (.5, .5)$

Figure 2: Agent type $v$ versus welfare (normalized by 100-ticket lottery outcome). On the left, in the 1-ticket lottery, many agents choose to take item 2 for certain rather than risk getting nothing. As a result, increasing $k$ benefits agents who weakly prefer item 1, at the expense of agents with stronger preferences. On the right, when $k \leq 2$, few agents spend a ticket securing item 2. This results in a waste trap: all agents benefit from increasing $k$.

Figure 3 presents a numerical study of such markets. We vary the supply of items $\mu$, as well as the distribution of preferences $F$. This figure demonstrates that the two regimes identified in Section 6.3 persist in markets with moderate supply. A tradeoff between values of $k$ occurs when a many agents are willing to spend tickets on less popular items: agents with the strongest preference for item 1 prefer $k = 1$, while those with more typical preferences prefer large $k$. Meanwhile, if supply is not too scarce and most agents have a strong preference for item 1, then small values of $k$ result in a waste trap, hurting all agents.

# 7 Practical Considerations

Although the $k$-ticket parallel lottery has many advantages in theory, it is natural to wonder whether they can be realized in practice. In this section, we discuss (i) when a parallel lottery is appropriate (ii) how it should be administered, and (iii) how the value of $k$ should be selected.




(a) $v \sim U[0,1]$ $\mu = (.5,.5)$

(b) $v \sim U[0,.4]$ $\mu = (.5,.5)$

(c) $v \sim U[0,1]$ $\mu = (.4,.4)$

(d) $v \sim U[0,.4]$ $\mu = (.4,.4)$

(e) $v \sim U[0,1]$ $\mu = (.3,.3)$

(f) $v \sim U[0,.4]$ $\mu = (.3,.3)$

(g) $v \sim U[0,1]$ $\mu = (.2,.2)$

(h) $v \sim U[0,.4]$ $\mu = (.2,.2)$

Figure 3: Agent type $v$ versus welfare (normalized by 100-ticket lottery outcome). In the left column, many agents have a weak preference for item 1, and do best when $k$ is large; agents with a strong preference for item 1 do best when $k = 1$. In the right column, all agents strongly prefer item 1. When item 2 is abundant, small values of $k$ result in a waste trap that hurts most or all agents. In both columns, as items become more scarce, the differences between mechanisms become insignificant.

## 7.1 Advantages of Random Serial Dictatorship

Random Serial Dictatorship (RSD) is frequently deployed in practice. This mechanism offers applicants a dominant strategy, eliminating the need for them to understand the level of competition for each item. Interestingly, RSD, CEEI, and the $k$-ticket parallel lottery have identical equilibrium outcomes if the supply of each item is proportional to the fraction of agents for whom that item is their first choice. Therefore, if items are similarly competitive, the practical advantages of RSD may outweigh any potential efficiency gains from more complex mechanisms.

## 7.2 Ensuring Agents Make Informed Decisions

If applicants have similar ordinal preferences but different preference intensities, the theoretical advantages of the $k$-ticket parallel lottery over RSD are greater. However, our analysis assumes equilibrium play. For the theoretical advantages of a parallel lottery to be realized in practice, applicants must understand their chances of winning each item.

Many agencies publish lottery statistics. For example, The Alaska Department of Fish and Wildlife publishes an annual Hunt Supplement, the National Park Service posts Half Dome Lottery statistics online,[8] and the New York City Department of Education provides an annual High School Directory. However, this information is typically released *after* the allocation process has concluded. Because items, quantities, and other features of the setting may change from year to year, this information is insufficient to allow agents to make good decisions.

Instead, we recommend the approach taken by the Singapore Housing and Development Board, which provides live updates on win probabilities using an online interface and allows agents to adjust their strategy until the final deadline.[9] A similar approach was used by Wake County Public Schools prior to 2015. The district used a non-strategy-proof mechanism, but displayed information about the number of students selecting each school as their first choice, and allowed students to react to this information in real time (Dur et al. (2018)).

In addition to deciding *when* to provide data, system administrators must decide *what* data to publish. Agencies often report the number of applications for each item. For $k = 1$, this information is adequate to determine success probabilities. For $k > 1$, however, this does not account for the fact

[8] `https://www.nps.gov/yose/planyourvisit/hdpermitsapps.htm`

[9] In theory, this approach could cause applicants to wait until just before the deadline to submit their preferences, and even to submit false preferences initially in order to affect others' choices. In practice, we expect that this will not be a major concern.

that some applicants will match to an item which they prefer. Similarly, the common approach of publishing the number of applicants that list each item as their first choice doesn't account for competition from applicants who rank the item second or third. A better solution is to use applicants' submitted preferences to report estimated success probabilities $p_i$.

When $k = 1$, an applicant who is informed of success probabilities for each item faces a fairly straightforward decision. For $k > 1$, the possibility of winning multiple items makes the decision problem more complex, as discussed in Section 3.1. For this reason, the best approach might be to offer an interface that allows users to enter a strategy (i.e., "4 tickets on item A, 1 on item B, and 1 on item C") and then displays an estimate of the associated allocation probabilities.

### 7.3 Favoring Certain Applicants

In some settings, it may be possible and desirable to distinguish between applicants, giving some an advantage over others. This can be accomplished by using parallel lotteries in which applicants are given different numbers of tickets. While the resulting allocation will obviously no longer be envy-free, it will remain approximately efficient. More specifically, any equilibrium of a parallel lottery (with heterogeneous ticket allocations) is $e/(e-1)$-trade efficient. Furthermore, no waste reallocation of such an equilibrium can improve welfare of an agent given $k$ tickets to more than $1 + 1/k$ times their welfare in the original equilibrium.

### 7.4 Allocating Leftover Items

In many settings, items that are not awarded during the initial allocation are made available on a first-come, first-served basis. However, Proposition 3 implies that a wasteful allocation is sometimes necessary to achieve effective sorting. As a result, reallocating underdemanded items comes with a consequence: applicants who consider underdemanded items acceptable will use all of their tickets competing for more popular options. By committing not to reallocate leftover items, the designer can incentivize applicants to apply for less popular options and improve outcomes for agents with strong preferences for popular items.

### 7.5 When Should Agents With Strong Preferences Be Favored?

When demand significantly exceeds supply, our results establish that all $k$-ticket outcomes are similar. In this case, we recommend using a 1-ticket parallel lottery, due to the factors discussed in

Section 7.2: simplicity of the strategy space and the clear connection between action and outcome.

When supply and demand are more balanced, the findings in Sections 6.3 and 6.4 suggest that smaller choices of $k$ are preferred by agents with atypically strong preferences for popular items. It is important to note the "strength" of an agent's preference is determined by her *relative* value for each item (i.e. $v_2/v_1$), rather than her *absolute* values $v_1$ and $v_2$. To illustrate, consider a simple scenario where agent $A$ has $v_1^A = 10, v_2^A = 6$. Suppose agent $B$ has a stronger relative preference for item 1, with $v_1^B/v_2^B = 5$. There are a continuum of possible values for the pair $(v_1^B, v_2^B)$, but we highlight two possibilities that are relevant to practice:

1. **Strong preferences indicate disability.** Agent $B$ has $v_1 = 10$, but constraints outside of her control result in a low value for item 2 ($v_2 = 2$).

2. **Strong preferences indicate good outside options.** Agent $B$ agrees with agent $A$ about the quality of each item, but has access to an outside option of value 5. Therefore, the value of being matched is $v_1^B = 10 - 5 = 5$ for item 1, and $v_2^B = 6 - 5 = 1$ for item 2.

In the former scenario, the designer may wish to give agent $B$ a greater chance of receiving item 1. This could be justified on utilitarian grounds (it is more "efficient" for $B$ to get item 1 and $A$ to get item 2), or based on a notion of equity ($B$ should not receive lower utility on the basis of a disability). In the latter scenario, it is harder to make the case for giving item 1 to $B$. Although we cannot distinguish between these scenarios by observing the behavior of agent $B$, the designer might possess contextual information about which of these scenarios is more likely. As we illustrate in Section 8, this information could help to guide the choice of $k$.

In summary, we recommend choosing a mechanism according to the following procedure:

1. If items are similarly competitive or if communicating success probabilities with applicants is too difficult, use Random Serial Dictatorship.

2. Otherwise, the $k$-ticket parallel lottery may be suitable. Choose $k$ appropriately:

   (a) If demand far exceeds supply, use the 1-ticket parallel lottery.

   (b) Use a large $k$ to favor agents with modest preferences for popular items.

   (c) Use the smallest $k$ that avoids the waste trap to favor agents with strong preferences for popular items.

# 8 Applications

In our final section, we apply the advice from Section 7 to several domains. Our intent is to illustrate how our work could be applied to practice. We do not claim to possess the domain expertise required to make definitive recommendations.

## 8.1 Outdoor Recreation

The allocation of hunting permits is one of several settings in which access to natural resources is regulated in the name of conservation. Park administrators frequently impose quotas on popular hiking trails, campsites, and portage routes. In many cases, these quotas are filled on a first-come, first-served basis. However, when demand far exceeds supply, this can result in a chaotic and inefficient allocation, so a $k$-ticket parallel lottery may be more appropriate.

The next question is, how should $k$ be chosen? Let us start with the lottery for Alaska bison hunts. In 2018, each hunt received at least 100 applications for every available permit (ADFG, 2019). In this case, Theorem 1 suggests that any $k$-ticket equilibrium will be approximately envy-free and Pareto efficient. Section 7.2 discusses reasons that equilibrium play may be more plausible for the 1-ticket parallel lottery than its $k$-ticket counterparts, so a 1-ticket parallel lottery could be a good solution for this market.

When demand is more modest, the results in Section 6.3 suggest that there may be a tradeoff when choosing $k$, with smaller values of $k$ favoring applicants who strongly prefer popular items. In outdoor recreation settings, popular items are often the most accessible. Ease of access is likely to be especially important to families with young children, the elderly, disabled, and workers with limited vacation. Many of these groups likely consider only a small number of destinations to be feasible. By choosing a small value of $k$, the park service could encourage avid adventurers to apply to less-popular options, thereby expanding access to groups with less flexibility.

## 8.2 Public Housing

Singaporean housing lotteries invite the question of whether the parallel lottery could also be used to allocate housing in the United States. Because some units are likely to be significantly more desirable than others, a simple serial dictatorship might lead to inefficient outcomes. Furthermore, because demand significantly exceeds supply, waste is unlikely to be a concern. Both of these facts suggest that a $k$-ticket parallel lottery might be appropriate.

In the public housing domain, there are good arguments for and against favoring agents with strong preferences for particular units. On the one hand, an applicant who does not possess a car may only be able to accept a unit that is close to their workplace or along a public transit route. On the other, some applicants with good outside options may consider all but the best public housing units to be unacceptable. One might wish to accommodate the first group without accommodating the second. Although an anonymous mechanism cannot achieve this, it might be possible to use observable characteristics such as income to exclude or de-prioritize applicants with strong outside options. In addition, these characteristics could be used to allocate different numbers of tickets to different applicants, as discussed in Section 7.3.

### 8.3 School Choice

By law, every student must receive a school assignment. As a result, waste is not an option in the school choice setting: even if the $k$-ticket parallel lottery produces a wasteful assignment, remaining students must eventually be assigned a seat at a school with available seats. This reduces some of the downside risk from listing popular schools, implying that even students with only moderate preferences for these schools might apply for them.

This may be for the best: we suspect that the students with the strongest preferences for elite public schools are typically those with good outside options (i.e., the ability to attend private schools). Favoring these students would only exacerbate inequity. All in all, the parallel lottery does not seem like an ideal choice for this setting.

### 8.4 College Scholarships

Students often have the opportunity to apply for a scholarship when they apply for admission to college. We assume that almost all students prefer to receive a scholarship. However, while the scholarship is a minor consideration for some students, for others, it determines their ability to attend. If the application cost is low, many students will apply, and scholarships may not go to the students who need them most.

An alternative approach would be to treat admission with a scholarship separately from admission without a scholarship, and allow students to apply to at most one of these options. At first, the idea of forcing students who apply for scholarships to forgo the possibility of receiving a non-scholarship offer may seem repugnant. However, this policy should typically *increase* the number of low-income students attending college.

The intuition for this claim is as follows. We can think of current practice as a sort of serial dictatorship, where students are ranked according to some criterion (perhaps a test score, evaluation of an essay, or a lottery) with the highest scoring students receiving a scholarship and the next tier being offered admission without a scholarship. Meanwhile, our proposed alternative resembles a 1-ticket parallel lottery with two goods. Because applying for a scholarship lowers the chance of admission, students whose parents can pay full tuition will not ask for a scholarship, leaving the financial support for those who need it.

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# A Proofs from Section 3

## A.1 Transformation of $k$-Ticket Strategy Choice into Downward Recursive Portfolio Choice

*Proof of Lemma 1.* Fix a vector $p$ of win probabilities, and define a multiset $S$ of items that consists of $k$ copies of each item $i$. For each item $i$, we define the cardinal utility of $i$ to be $u_i := v_i$, and the success chance to be $\alpha_i := p_i$. Let $\succ_v$ be a total ordering on $S$ according to the true preferences of an agent type $v$, with ties between identical items broken in a consistent manner.

The Marginal Improvement Algorithm (MIA) of Chade and Smith (2006) determines a portfolio $\hat{Y} \subseteq S$ that maximizes the gross payoff

$$f(Y) = \sum_{i \in Y} \Big( \alpha_i u_i \prod_{j \in Y \mid j \succ_v i} (1 - \alpha_j) \Big) = \sum_{i \in Y} \Big( p_i v_i \prod_{j \in Y \mid j \succ_v i} (1 - p_j) \Big) \tag{12}$$

for an agent type $v$. Let $\succ$ be a preference ordering of $[n]$ consistent with $\succ_v$, and let $t = (t_1, t_2, ..., t_n)$ be the ticket choice vector constructed by setting each $t_i$ equal to the number of copies of item $i$ in $\hat{Y}$. Expanding (12), we have

$$f(\hat{Y}) = \sum_{i \in [n]} \Big( v_i (1 - (1 - p_i)^{t_i}) \prod_{j \succ i} (1 - p_j)^{t_j} \Big) = v \cdot \chi((t, \succ), p) = w(v, (t, \succ), p). \tag{13}$$

Thus, given $v$, the MIA determines a strategy $(t, \succ)$ that maximizes $w(v, (t, \succ), p)$.

Let $s_Y$ denote the strategy consisting of the ticket choice vector corresponding to $Y \subseteq S$ and a preference order $\succ$ reflecting the true preferences of a fixed agent type $v$. On our portfolio, the MIA proceeds as follows:

1. Begin with an empty portfolio $Y_0 = \emptyset$.

2. Choose any item

$$i \in \arg\max_{i \in S \setminus Y_{j-1}} w(v, s_{Y_{j-1}+\{i\}}, p). \tag{14}$$

3. If

$$w(v, s_{Y_{j-1}+\{i\}}, p) - w(v, s_{Y_{j-1}}, p) > c(n), \tag{15}$$

   set $Y_j := Y_{j-1} + \{i\}$ and go to Step 2.

Our algorithm is equivalent to the MIA with the additional requirement that the process of strategy augmentation stop after $k$ steps. This is ensured by defining a cost function $c$ such that $c(|Y|) = 0$ if $Y \leq k$ and $c(|Y|) = \infty$ if $|Y| > k$ (referred to in Chade and Smith (2006) as the *fixed sample size* $k$ case). The optimality of the greedy algorithm follows from the optimality of the resulting instance of downward recursive portfolio choice. □

## A.2 Existence of Equilibria

In this section, we complete the proofs that every strategy profile $\Phi$ implies a unique consistent win probability vector $p$ (Lemma 2) and that every $k$-ticket lottery has an equilibrium (Proposition 1).

*Proof of Lemma 2.* Given a strategy profile $\Phi$, our setting maps directly to the two-sided matching market defined by Azevedo and Leshno (2016). In their framework, agent types are identified by a preference ranking of items $\tilde{\succ}$ and a score $e_i$ that determines priority for each acceptable item. Each strategy $s = (t, \succ)$ in our model corresponds to a measure over types $(\tilde{\succ}, e)$ as follows:

1. $\tilde{\succ}$ ranks items with $t_i > 0$ according to $\succ$.
2. items with $t_i = 0$ are unacceptable according to $\tilde{\succ}$.
3. $e_i$ is distributed as the first order statistic of $t_i$ uniform random draws from $[0, 1]$. Scores are independent across items.

Allocation in Azevedo and Leshno (2016) is summarized by a vector of cutoffs that lists the minimum score required to receive each item $i$. Under this mapping, the vector $c$ of market clearing cutoffs is precisely $\vec{1} - p$, and the consistency condition is equivalent to the market clearing condition (Definition 2 of Azevedo and Leshno (2016).)

Corollary A1 in the appendix of Azevedo and Leshno (2016) implies that we can find a set of market clearing cutoffs $c$ (equivalently, a vector of win probabilities $p$). Theorem A1 implies that the set of market clearing cutoffs $c$ forms a lattice with the supremum and infimum operators in $[0, 1]^n$. By Theorem A2, the measure of agents matched to each item $i$ is the same under any market clearing $c$ (consistent $p$).

For each item $i$, the distribution of scores $e_i$ has full support for each strategy $s$ that places at least one ticket in the lottery for item $i$. As a result, any isolated decrease in a win probability for an item that a positive measure of agents receive results in strictly fewer agents matching. Thus the minimal and maximal elements of the lattice of consistent $c$ coincide, implying that it contains a unique element. □

*Proof of Proposition 1.* Fix $k$, a set of items, an agent mass, and an agent type measure. We define a function $f : [0,1]^n \to [0,1]^n$ as follows. For any win probability vector $p$, denote by $\Phi^p$ the strategy profile determined by Lemma 1 that is unique up to a set of agents with measure zero. We define $f(p)$ to be the vector of win probabilities determined by $\Phi^p$, which is unique by Lemma 2.

By inspection, a fixed point of $f$ corresponds to a $k$-ticket equilibrium. We show that $f$ is continuous, from which the existence of a fixed point follows from Brouwer's theorem.

For $s \in S_k$, we have

$$(\Phi^p)^{-1}(s) = \{v : \forall s' \in S_k \setminus \{s\},\ v \cdot \chi(s,p) > v \cdot \chi(s',p)\}. \tag{16}$$

Because $\chi(s,p)$ is constant in $v$, $(\Phi^p)^{-1}(s)$ is an open convex polytope in $\mathbb{R}^n$ (or the empty set).

Let $(j)$ denote the item on which a given agent $v$ uses her $j^{th}$ ticket according to the algorithm specified in Lemma 1. Lemma 1 implies that the win probability of the items on which $v$ bids is weakly decreasing. Using this fact to unwind the algorithm, we have that $\Phi^p(v)$ is determined by the unique sequence of items $(1),(2),...,(k)$ such that $\forall i \in [n], j \in [k]$,

$$\sum_{l=1}^{j} \big(v_{(l)} p_{(l)} \prod_{m=l+1}^{j} (1 - p_{(m)})\big) \geq v_i p_i + (1-p_i)\Big(\sum_{l=1}^{j-1} \big(v_{(l)} p_{(l)} \prod_{m=l+1}^{j-1} (1 - p_{(m)})\big)\Big). \tag{17}$$

Because each constraint is polynomial in $p$, and because $\eta$ is continuous, $\eta((\Phi^p)^{-1}(s))$ is continuous in $p$ for all $s$.

Let $X(f(p))$ be the matrix in which each column is the probabilistic allocation for a strategy under $f(p)$, that is,

$$X_{(i,s)}(f(p)) := \chi_i(s, f(p)), \tag{18}$$

for all $i \in [n]$, $s \in S_k$. Let $H(p)$ be the vector containing the measure of agents who adopt each strategy under $\Phi^p$, that is,

$$H_s(p) := \eta((\Phi^p)^{-1}(s)), \tag{19}$$

for all $s \in S_k$. If in fact $f(p)_i < 1$ for all $i \in [n]$, $f(p)$ is the unique vector which satisfies

$$X(f(p)) \cdot H(p) = \mu, \tag{20}$$

as this equation is equivalent to the consistency constraints. However, the consistency constraints allow the possibility that some items go unallocated if $f(p)_i = 1$. We generalize (20) by defining the map $A$ as follows.

$$A_i(f(p), p) := \Big(\sigma\big(X_i(f(p))H(p) - \mu_i\big) + (1 - f_i(p))\Big)\Big(X_i(f(p))H(p) - \mu_i\Big), \tag{21}$$

where $\sigma(y) = \max(0, y)$. By inspection, $A(f(p), p) = 0$ if and only if $f(p)$ is consistent with $\Phi^p$.

$A(f(p), p)$ is continuous in $f(p)$ and $p$, and by Lemma 2, $p$ determines a unique $f(p) \in [0, 1]^n$ such that $A(f(p), p) = 0$. Thus $f(p)$ is continuous in $p$, and an equilibrium point exists by Brouwer's theorem. □

### A.3 1-Ticket Lotteries Have Unique Equilibria

The following lemma demonstrates that 1-ticket lotteries have a unique win probability vector $p$. Because $p$ is unique, every agent receives the same expected value in every equilibrium. Note that the result is very similar to a result of Gale (1976), who considered the case where agents may receive more than one item.

**Lemma 3** (Uniqueness of 1-ticket Equilibria)**.** *if $(\Phi, p)$ and $(\Phi', p')$ are 1-ticket equilibria, then $p = p'$.*

*Proof.* Let $(\Phi, p)$ and $(\Phi', p')$ be 1-ticket equilibria. For contradiction, assume $p \neq p'$. Without loss of generality, for some item $i$, we have $p_i > p'_i$. Because every agent has a single ticket, we can write a closed-form expression for $p_i$. Let $t(v)$ denote the ticket choice vector of agent type $v$ according to $\Phi$ and $t'(v)$ denote the ticket choice vector of agent type $v$ according to $\Phi'$. For each item $i$, we have

$$p_i = \min\{1, \frac{\mu_i}{\eta(\{v : t_i(v) = 1\})}\}, \tag{22}$$

and the analogous statement holds for $t'(v)$.

Thus $p_i > p'_i$ implies the existence of a set of agents $V$ with $\eta(V) > 0$ such that for every $v \in V$, $t_i(v) = 0$ and $t'_i(v) = 1$. Let $j$ be the item that maximizes $\eta(\{v \in V : t_j(v) = 1\})$. Because agents in the set $\{v \in V : t_j(v) = 1\}$ switched from $j$ to $i$, it must be the case that for all $v \in V$,

$$p_j v_j > p_i v_i > p'_i v_i > p'_j v_j. \tag{23}$$

Rearranging, we have

$$\frac{p_j}{p'_j} > \frac{p_i}{p'_i}. \tag{24}$$

This corresponds to the intuition that if item $i$ is relatively fully allocated in $(\Phi', p')$ compared to $(\Phi, p)$, some other item $j$ must be more relatively fully allocated in $(\Phi', p')$ to create additional demand for item $i$. Repeating this argument implies the existence of a sequence of items, each more relatively fully allocated than the last. Because the number of items is finite, this presents a contradiction, and thus $p = p'$. □

## A.4 $k$-Ticket Lotteries Are Not Unique for $k > 1$

Although every 1-ticket parallel lottery has a unique equilibrium, $k$-ticket lotteries do not necessarily have unique equilibria when $k$ is greater than 1. The following example demonstrates a 2-ticket parallel lottery with multiple equilibria.

Consider a lottery with two items and $\mu_1 = \mu_2 = 1$. For simplicity, we consider a setting with two discrete groups of agents.[10] The first group of agents has type $(1, 0)$. Regardless of other agents' behaviors, these agents will always place both of their tickets in the lottery for item 1. The mass of this group is $1/(2\epsilon - \epsilon^2)$, which ensures that if only these agents apply for item 1, $p_1 = \epsilon$. We set $\epsilon > 0$ to be small enough that the win probability $p_1$ is (multiplicatively) close to $\epsilon$ regardless of the behavior of other agents. The second group of agents has type $(\frac{7}{32\epsilon}, 1)$ and mass 4/3. This scenario is summarized in Table 1.

| Agent Group | Mass | Type |
|---|---|---|
| Group 1 | $1/(2\epsilon - \epsilon^2)$ | $(1, 0)$ |
| Group 2 | 4/3 | $(\frac{7}{32\epsilon}, 1)$ |

Table 2: A 2-ticket parallel lottery that yields 2 equilibria.

In the first equilibrium, group 1 agents enter both tickets in the lottery for item 1 and group 2 agents put both tickets in the lottery for item 2. The resulting vector of win probabilities is $p = (\epsilon, 1/2)$. In this equilibrium, a group 2 agent receives expected values of $14/32 - 7\epsilon/32$ from

[10]This assumption can be relaxed by replacing the agent point masses with Gaussians distributed tightly around the original points.

putting both tickets in the lottery for item 1, 24/32 from putting both tickets in the lottery for item 2, and $23/32 - \epsilon/2$ from splitting her tickets, so the strategy profile is optimal given $p$.

In the second equilibrium, group 1 agents enter both tickets in the lottery for item 1 and group 2 agents split their tickets between the two lotteries. The resulting vector of win probabilities is $p = (\epsilon', \frac{3}{4-4\epsilon'})$, for some $\epsilon' < \epsilon$ with $\epsilon' \approx \epsilon$. In this equilibrium, a group 2 agent receives expected values of $\frac{\epsilon'}{\epsilon}(14/32 - 7\epsilon'/32) \approx 14/32$ from putting both tickets in the lottery for item 1, $\frac{24(1-\epsilon')-9}{16(1-\epsilon')^2} \approx 30/32$ from putting both tickets in the lottery for item 2, and $\frac{\epsilon'}{\epsilon}(7/32) + 3/4 \approx 31/32$ from splitting her tickets, so the strategy profile is optimal given $p$.

# B Proof of Theorem 1

In this section, we prove that for almost every agent, the ratio of their expected value at any $k$-ticket equilibrium to their expected value under CEEI goes to 1 as the mass of agents increases. The proof treats $\mu$, $\eta$, and $k$ as fixed. When $x$ is an allocation corresponding to a specified $k$-ticket equilibrium, we write $p^x$ to denote the vector of win probabilities at this equilibrium.

We begin by proving two properties of equilibria which hold for large agent masses. First, for sufficiently large $M$, we prove that the maximum and minimum win probabilities for each item are $\Theta(1/M)$ at equilibrium. Second, we show that as demand for each item increases, the $\eta$-measure of the set of agents who bid on more than one item goes to 0. As a result, as $M$ increases, the ratio of expected value at any $k$-ticket equilibrium to expected value at the unique 1-ticket equilibrium goes to 1 for almost every agent. Finally, we observe that the 1-ticket parallel lottery produces the same allocation as CEEI when all items are fully allocated.

**Claim 1** (Equilibrium win probabilities are $\Theta(1/M)$)**.** *Let $\mu$, $\eta$, and $k$ be fixed as above. There exist positive constants $c_1, c_2$ such that for every $M \geq max_j \mu_j$, every $x \in X_M$, and every $i \in [n]$,*

$$\frac{c_1}{M} \leq p_i^x \leq \frac{c_2}{M}. \tag{25}$$

*Proof.* Fix $x \in X_M$. We have for each item $i$ that

$$\frac{\mu_i}{kM} \leq p_i^x. \tag{26}$$

This follows immediately from the fact that at most $kM$ tickets can possibly be entered into a single lottery, and at least $\mu_i$ of these are drawn if $p_i^x < 1$.

We prove the upper bound by inductively constructing a set of items with win probabilities bounded by constant multiples of $1/M$. First, observe that in any equilibrium, there exists an item $i_1$ such that $p_{i_1}^x \leq \frac{\sum_i \mu_i}{M}$. Otherwise, the quantity of items allocated by $x$ would exceed $\sum_i \mu_i$, violating the equilibrium requirements. Furthermore, by assumption, some agents have nonzero value for a second item $j$. Thus there exist constants $1 > \epsilon_1, \gamma_1 > 0$ such that

$$\eta(\{v : \exists j \neq i_1, v_j > \epsilon_1 v_{i_1}\}) > \gamma_1. \tag{27}$$

As a result, there exists an item $i_2$ such that $p_{i_2}^x \leq \frac{\sum_i \mu_i}{\epsilon_1 \gamma_1 M}$. Otherwise, by Lemma 1, a mass $\gamma_1 M$ of agents would place their first ticket on items other than $i_1$. Observe that this ensures each agent is allocated with probability at least $\frac{\sum_i \mu_i}{\epsilon_1 \gamma_1 M}$ in the 1-ticket case, and the total allocation probability of each agent is strictly increasing in the number of tickets when $p^x$ is held constant. This presents a contradiction, as the quantity of items allocated to the set of agents with mass $\gamma_i M$ would exceed $\sum_i \mu_i$. Repeating this argument for $m = 2, 3, ..., n$, we construct a series of constants $1 > \epsilon_m, \gamma_m > 0$ such that

$$\eta(\{v : \exists j \neq i_1, ..., i_{m-1}, v_j > \epsilon_m \max\{v_{i_l}\}_{l=1}^{m-1}) > \gamma_m, \tag{28}$$

and a series of items $i_m$ such that

$$p_{i_m}^x \leq \frac{\sum_{i=1}^n \mu_i}{M \prod_{j=1}^{m-1} \epsilon_j \gamma_j}. \tag{29}$$

□

**Claim 2** (The set of ticket splitters goes to 0)**.** *For $x \in X_M$, let $S_x$ be the set of ticket splitters, agents whose optimal strategy under Lemma 1 bids on more than one item. We have*

$$\lim_{M \to \infty} \sup_{x \in X_M} \eta(S_x) = 0. \tag{30}$$

*Proof.* By definition, every agent in $S_x$ adopts an equilibrium strategy in which she bids on at least two items. Let $v$ be an agent in $S_x$ who places her first $t-1$ tickets in the lottery for item $i$ and her $t^{th}$ ticket in the lottery for item $j$. By Lemma 1, we have

$$p_i^x v_i > p_j^x v_j. \tag{31}$$

$$p_j^x v_j + (1-p_j^x)(1-(1-p_i^x)^{t-1})v_i > p_i^x v_i + (1-p_i^x)(1-(1-p_i^x)^{t-1})v_i. \tag{32}$$

Rearranging yields

$$\frac{p_i^x}{p_j^x} > \frac{v_j}{v_i} > \frac{p_i^x}{p_j^x} + (1-\frac{p_i^x}{p_j^x})(1-(1-p_i^x)^{t-1}), \tag{33}$$

and making use of the fact that $(1-(1-p_i^x)^{t-1}) \leq kp_i^x$ for $t-1 < k$, we have

$$S_x \subseteq \{v : \frac{v_i}{v_j} \in [\frac{p_i^x}{p_j^x}(1-k(p_i^x - p_j^x)), \frac{p_i^x}{p_j^x}]\}. \tag{34}$$

Claim 1 implies $p_i^x/p_j^x$ is upper-bounded by a constant, so Claim 2 follows from the absolute continuity of $\eta$ as the interval shrinks to zero. □

**Lemma 4** ($k$-ticket utility converges to 1-ticket utility)**.** *Let $x_M^1$ denote the allocation corresponding to the unique 1-ticket equilibrium with agent mass $M$. For almost every agent $v$, we have*

$$\lim_{M\to\infty} \sup_{x\in X_M} \frac{x(v)\cdot v}{x_M^1(v)\cdot v} = \lim_{M\to\infty} \inf_{x\in X_M} \frac{x(v)\cdot v}{x_M^1(v)\cdot v} = 1. \tag{35}$$

*Proof.* To prove Lemma 4, we must understand the structure of the 1-ticket parallel lottery. Consider the outcome of a 1-ticket parallel lottery in a market with parameters $\mu$, $\eta$, and $M$. By Lemma 3, this 1-ticket parallel lottery has a unique equilibrium win probability vector $p$ which in turn determines a unique expected value for any agent that plays an optimal strategy. Suppressing the dependence on $\eta$, we write $w^1(\mu, M, v)$ to denote the expected value of an agent $v$ at 1-ticket equilibrium. We proceed to show that $w^1$ is continuous in $\mu$ and $M$.

Let $V_i(p)$ denote the measure of the set of agents who bid on good $i$ under the optimal strategy given by Lemma 1 when the win probability vector is $p$. In the 1-ticket parallel lottery, for all $i$, we have

$$\mu_i = Mp_iV_i(p) \tag{36}$$

when $M$ is sufficiently large. By Lemma 3, each agent $v$ bids on a good $i \in \arg\max_{i\in[n]} v_i p_i$ at equilibrium. $V_i(q)$ is thus continuous in $p$, as it is the measure of a polytope whose sides are defined by linear equations in $p$. As a result, $\mu$ is continuous in $p$ in the 1-ticket lottery. We write $\mu^1(p) := \mu^1(p, M, \eta)$ to indicate this.

By Lemma 3, $\mu$ determines a unique value of $p$ satisfying the system of equations defined by (36). Combined with the fact that $\mu$ is continuous in $p$, this implies that $p$ is a continuous function of $\mu$. For fixed $\mu$ and sufficiently large $M$, scaling $M$ by a constant $c$ means that (36) is satisfied by the win probability vector $p/c$,[11] and thus $p$ is a continuous function of $M$ as well. Finally, the expected value of any agent $v$ at 1-ticket equilibrium is $\max_{s\in S_1} w(v, s, p)$, which is a continuous function of $p$. Thus $w^1$ is continuous in $\mu$ and $M$.

Define $i(p, v) := \arg\max_{i\in[n]} v_i p_i$.[12] By Lemma 1, the optimal strategy for an agent $v$ is to place her first ticket in the lottery for good $i(p, v)$, after which the marginal benefit for each additional ticket weakly decreases. Thus for all $M$, $x \in X_M$, we have

$$(1 - (1 - p^x_{i(p,v)})^k) v_{i(p,v)} \le x(v) \cdot v \le k p^x_{i(p,v)} v_{i(p,v)} = w^1(M, \mu^1(kp^x), v). \tag{37}$$

In (37), the first inequality follows because $x(v) \cdot v$ is a weakly better strategy than placing all $k$ tickets in the lottery for good $i(p, v)$, the second inequality follows because $x(v) \cdot v$ is weakly less than $k$ times the marginal benefit of the first ticket placed, and the final equality follows because $k p^x_{i(p,v)} v_{i(p,v)}$ is exactly the expected value of $v$ in a 1-ticket lottery with win probability vector $kp^x$.

From this observation and Claim 1 it follows that

$$\lim_{M\to\infty} \sup_{x\in X_M} \frac{x(v) \cdot v}{w^1(M, \mu^1(kp^x), v)} = \lim_{M\to\infty} \inf_{x\in X_M} \frac{x(v) \cdot v}{w^1(M, \mu^1(kp^x), v)} = 1. \tag{38}$$

As $w^1(M, \mu, v) = x^1_M(v) \cdot v$, it remains to prove that

$$\lim_{M\to\infty} \sup_{x\in X_M} \frac{w^1(M, \mu^1(kp^x), v)}{w^1(M, \mu, v)} = \lim_{M\to\infty} \inf_{x\in X_M} \frac{w^1(M, \mu^1(kp^x), v)}{w^1(M, \mu, v)} = 1. \tag{39}$$

[11] Note that $V_i(p) = V_i(cp)$ as long as $cp_i \le 1$ for all $i \in [n]$.

[12] Given $p$, we will ignore the set of agents who have multiple optimal strategies and for whom $i(p, v)$ is undefined, as this set has measure 0 and will not affect equilibrium parameters.

As $w^1$ is continuous in $\mu$ and $M$, it is sufficient to show that for all $i \in [n]$,

$$\lim_{M\to\infty} \sup_{x\in X_m} \frac{\mu_i^1(kp^x)}{\mu_i} = \lim_{M\to\infty} \inf_{x\in X_m} \frac{\mu_i^1(kp^x)}{\mu_i} = 1. \tag{40}$$

For sufficiently large $M$, we have

$$\mu_i^1(kp^x) = Mkp^x V_i(p^x) \tag{41}$$

by (36). Moreover, for any $M$, and any $x \in X_M$, $i \in [n]$, we have

$$M(1-(1-p_i^x)^k)V_i^k(p^x) \le \mu_i \le M(1-(1-p_i^x)^k)V_i^1(p^x), \tag{42}$$

where $V_i^k(p^x)$ denotes the measure of the set of agents whose optimal strategy under $p^x$ is to place all $k$ tickets into the lottery for item $i$, and $V_i^1(p^x)$ denotes the measure of the set of agents whose optimal strategy is to place at least 1 ticket in the lottery for item $i$. Equation (42) holds because the lefthand side is the mass of item $i$ allocated to agents who place every ticket in the lottery for item $i$ and the righthand side is the amount of item $i$ which would be distributed if every agent who bid on item $i$ placed every ticket in the lottery for item $i$.

Dividing (41) by (42) yields

$$\lim_{M\to\infty} \sup_{x\in X_m} \frac{kp_i^x}{(1-(1-p_i^x)^k)} \le \lim_{M\to\infty} \sup_{x\in X_m} \frac{\mu_i^1(kp^x)}{\mu_i} \le \lim_{M\to\infty} \sup_{x\in X_m} \frac{kp_i^x}{(1-(1-p_i^x)^k)} \frac{V_i^1(p^x)}{V_i^k(p^x)} \tag{43}$$

for sufficiently large $M$. The expressions on left and right both evaluate to 1, as by Claim 1 we have $\lim_{M\to\infty} \sup_{x\in X_m} kp^x/(1-(1-p_i^x)^k) = 1$, and $\lim_{M\to\infty} \sup_{x\in X_m} V_i^1(p^x)/V_i^k(p^x) = 1$ by Claim 2. The analogous claim for the infimum holds by identical reasoning, so (40) follows. □

**Lemma 5** (1-ticket lotteries converge to CEEI)**.** *Let $x_M^1$ be an allocation corresponding to a 1-ticket equilibrium where $p_i < 1$ for all $i \in [n]$. For almost every agent $v$, $x_M^1(v) = ce_M(v)$.*

*Proof.* A market defined by $\mu$, $\eta$ and $M \ge \sum_i \mu_i$ corresponds to the instance of CEEI in which the set of items is $[n] + \{o\}$, where $o$ indicates an outside option with value 0 and quantity $M - \sum_i \mu_i$.

When each agent is given a budget of 1, the price vector which assigns the price $1/p_i^{x_M^1}$ to each item and 0 to the outside option clears the market. To see this, observe that each agent has sufficient budget to buy a probability share $p_i^{x_M^1}$ in any item $i \in [n]$. For almost every agent $v \in V$, there exists a unique item $i \in K$ that maximizes $p_i^{x_M^1} v_i$, and spending her entire budget on this item maximizes her expected value. Thus almost every agent purchases a share $p_i^{x_M^1}$ in the same item on which she bids in the 1-ticket equilibrium, and picks up a share $1 - p_i^{x_M^1}$ of the outside option for free. By Lemma 1, this allocation is equivalent to $x_M^1(v)$. □

Theorem 1 follows directly from Lemmas 4 and 5, as every item is fully allocated for sufficiently large values of $M$.

# C Proofs from Section 5

## C.1 $k$-Ticket Lotteries are Ex-Post Trade Efficient

We say that an allocation is ex post trade efficient if agents cannot mutually benefit from ex post trades of their received items. Formally, we define the *envy graph* of an allocation $x$ to be a directed graph on $[n]$ containing a directed edge $i \to j$ for every $i, j$ such that $x_i(v) > 0$ and $v_i < v_j$ for every $v$ in a set $V_{i\to j}$ with $\eta(V_{i,j}) > 0$. The edge $i \to j$ in the envy graph indicates that a set of agents with nonzero measure will receive item $i$ and would benefit from an ex post trade for item $j$.

**Definition 10.** *An allocation $x$ is* ***ex post trade efficient*** *if its envy graph is acyclic.*

**Lemma 6.** *Ex ante trade efficiency implies ex post trade efficiency.*

*Proof.* We prove the contrapositive. Consider an allocation $x$ whose envy graph contains a cycle $C$. Define

$$capacity(C) := \min_{i\to j \in C} \int_{V_{i\to j}} x_i(v)\, d\eta, \tag{44}$$

and note that $capacity(C) > 0$ by definition. For each edge $i \to j$ in $C$, choose a subset $V'_{i\to j} \subseteq V_{i\to j}$ such that $\int_{V'_{i\to j}} x_i(v)\, d\eta = capacity(C)$ and exchange $i$ for $j$ to create an allocation that Pareto dominates $x$ and keeps $\int x(v)\, d\eta$ constant. □

**Theorem 6.** *Any $k$-ticket parallel lottery outcome is ex post trade efficient.*

*Proof.* Consider the envy graph of an allocation $x$ corresponding to a $k$-ticket equilibrium. If the envy graph contains $i \to j$, then there exists a set $V_{i\to j}$ with $\eta(V_{i\to j}) > 0$ such that for every $v \in V_{i\to j}$, $x_i(v) > 0$ and $v_i < v_j$. This implies $p_i > p_j$, as no agent in $V_{i\to j}$ places a ticket in the lottery for item $i$ otherwise. Thus a cycle in the envy graph would correspond to a cycle of strict inequalities in win probabilities, which is impossible. □

## C.2 $k$-Ticket Lotteries are Not (Ex-Ante) Trade Efficient

**Proposition 6.** *For $k > 1$, if $\eta$ has full support on $\mathbb{R}^n$, any equilibrium with goods $i$ and $j$ such that $p_j < p_i < 1$ corresponds to an allocation that is not trade efficient.*

*Proof.* Let $V_{i,j}$ denote the set of agent types $v$ such that

$$p_i v_i \geq p_j v_j \tag{45}$$

$$p_j v_j + (1 - p_j) p_i v_i \geq p_i v_i + (1 - p_i) p_i v_i \tag{46}$$

$$\forall l \notin \{i, j\}, v_l < 0. \tag{47}$$

By Lemma 1, agents in $V_{i,j}$ place their first ticket in the lottery for good $i$ and their remaining tickets in the lottery for good $j$. Solving the two equations, we find that

$$V_{i,j} = \{v : \frac{p_j v_j}{p_i v_i} \in [1 - (p_i - p_j), 1], v_l < 0 \text{ for } l \notin \{i, j\}\}. \tag{48}$$

Because $\eta$ has full support, $\eta(V_{i,j}) > 0$, and $V_{i,j}$ contains sets of agents with nonzero measure who have different relative values for items $i$ and $j$ and a chance of receiving nothing. In this case, there exists an exchange rate at which agents benefit from exchanging shares of $i$ and $j$. □

## C.3 Upper Bounds on Trade Efficiency

Consider a $k$-ticket equilibrium defined by a win probability vector $p$ and an allocation $x$. Let $y$ be a trade reallocation of $x$, and let $c_i := C(p_i)$, for a decreasing cost function $C : [0, 1] \to \mathbb{R}_+$ defined later. Without loss of generality, we assume

$$0 \leq p_1 \leq p_2 \leq \cdots \leq p_n \leq 1. \tag{49}$$

Because $y$ is a trade reallocation of $x$, we have

$$\int c \cdot x(v) \, d\eta = \int c \cdot y(v) \, d\eta.$$

Thus there exists a set $V$ with $\eta(V) > 0$ such that for each $v \in V$, $c \cdot y(v) \leq c \cdot x(v)$. Fix $v \in V$, and define

$$w_i := p_i v_i, \tag{50}$$

the utility of placing a single ticket on item $i$. For $j \in \{0, ..., k\}$, let $u_j$ be the welfare of agent $v$ corresponding to the optimal strategy with $j$ tickets. Thus $u_0 = 0$ and for $j > 0$, Lemma 1 implies

$$u_j = u_{j-1} + \max_{i \in [n]} (w_i - p_i u_{j-1}). \tag{51}$$

Let $i_j$ be the item on which the $j^{th}$ ticket is placed. (If there are two or more strategies consistent with $v$, we break ties in favor of the item with the lowest index). As $u_j$ is increasing in $j$, it follows from (51) that

$$p_{i_1} \geq p_{i_2} \geq \cdots \geq p_{i_k}. \tag{52}$$

Define

$$\delta_j := u_j - u_{j-1} \tag{53}$$

to be the utility gain from the $j^{th}$ ticket, and note that

$$v \cdot x = u_k = \sum_j \delta_j. \tag{54}$$

By (51), we have that for all $i \in [n]$, $j \in [k]$,

$$w_i - p_i u_{j-1} \leq w_{i_j} - p_{i_j} u_{j-1}. \tag{55}$$

Rearranging yields the following set of incentive compatibility constraints: for all $i \in [n]$, $j \in [k]$,

$$w_i \leq w_{i_j} + p_i u_{j-1} - p_{i_j} u_{j-1} = u_j - u_{j-1} + p_i u_{j-1} = \delta_j + p_i u_{j-1}. \tag{56}$$

Inspired by this, we define $W : [0, 1] \to \mathbb{R}_+$ by

$$W(p) = \min_{j \in [k]} (u_j - u_{j-1} + p u_{j-1}) = \min_{j \in [k]} (\delta_j + p u_{j-1}), \tag{57}$$

which ensures that for all $i \in [n]$,

$$w_i \leq W(p_i), \tag{58}$$

and allows us to prove a useful bound on welfare after trading.

**Lemma 7.** *For any $p \in [0,1]^n$ satisfying (49) and any $v \in \mathbb{R}^n_+$, define $W$ as in (57). For any cost function $C : [0,1] \to \mathbb{R}_+$, define $c_i := C(p_i)$. For any pair $(\Delta, U)$ such that*

$$pC(p)\Delta + pU \geq W(p) \quad \forall p \in [0,1], \tag{59}$$

*and any allocation $y$ such that $y \cdot \mathbf{1} \leq 1$, the following holds:*

$$v \cdot y \leq \Delta(c \cdot y) + U. \tag{60}$$

*Proof.* For any $B \geq 0$, if we take the dual of the linear program that maximizes $v \cdot y$ subject to the constraints

$$c \cdot y \leq B,$$
$$y \cdot \mathbf{1} \leq 1,$$

we get the following program with two variables and $n$ constraints:

$$\begin{aligned} \min_{\Delta, U \geq 0} \quad & B\Delta + U \\ s.t. \quad & c_i p_i \Delta + p_i U \geq p_i v_i = w_i \quad \forall i. \end{aligned} \tag{61}$$

Equation (58) implies that (59) is stronger than (61). Weak duality thus implies (60) for any $(\Delta, U)$ satisfying (59). □

### C.3.1 Proof of Theorem 3

We begin by introducing new parameters for the proof. Define $q_0 = 1$, $q_k = 0$, and for $j \in [k-1]$ define

$$q_j := 1 - \frac{\delta_{j+1}}{\delta_j}. \tag{62}$$

Note that $W$ is the minimum over $k$ linear functions and that $q_j$ is the point at which functions $j$ and $j+1$ intersect; i.e., $q_j$ solves

$$\delta_j + qu_{j-1} = \delta_{j+1} + qu_j.$$

The following Lemma establishes useful properties of $\{q_j\}_{j\in[k-1]}$.

**Lemma 8.** *For any $p \in [0,1]^n$ satisfying (52) and $w \in \mathbb{R}_+^n$, define $\{u_j\}_{j=0}^k$ as in (51), $\{\delta_j\}_{j=1}^k$ as in (53) and $\{q_j\}_{j=0}^k$ as in (62). We have*

$$0 = q_k \leq q_{k-1} \leq \cdots \leq q_1 \leq q_0 = 1. \tag{63}$$

*Moreover, for any $\alpha \in [0,1]$ and $m \in \arg\min_{j\in[k]} \delta_j + \alpha u_{j-1}$,*

$$q_m \leq \alpha \leq q_{m-1}. \tag{64}$$

*Proof.* From combining (51) and (56), we observe that

$$\begin{aligned}\delta_{j+1} = w_{i_{j+1}} - p_{i_{j+1}} u_j \geq w_{i_j} - p_j u_j \geq \delta_j(1 - p_{i_j}),\\ \delta_j = w_{i_j} - p_{i_j} u_{j-1} \geq w_{i_{j+1}} - p_{i_{j+1}} u_{j-1} = \delta_{j+1} + p_{i_{j+1}} \delta_j.\end{aligned}$$

Rearranging gives

$$p_{i_j} \geq 1 - \frac{\delta_{j+1}}{\delta_j} = q_j \geq p_{i_{j+1}},$$

and (63) follows from (52).

For $j \in [k]$, define $L_j(\alpha) = \delta_j + \alpha u_{j-1}$ to be the $j^{th}$ line in the set $\{\delta_j + \alpha u_{j-1}\}_{j\in[k]}$. Then $q_j$ is the point where $L_j$ and $L_{j+1}$ intersect, and

$$L_j(\alpha) \leq L_{j+1}(\alpha) \Leftrightarrow \alpha \geq q_j. \tag{65}$$

Suppose that (64) holds. Because $\alpha \geq q_m \geq q_{m+1} \geq \cdots \geq q_k$, (65) implies

$$L_m(\alpha) \leq L_{m+1}(\alpha) \leq \cdots \leq L_k(\alpha).$$

Similarly, $\alpha \leq q_{m-1} \leq q_{m-2} \leq \cdot \leq q_0$ implies

$$L_m(\alpha) \leq L_{m-1}(\alpha) \leq \cdots \leq L_1(\alpha).$$

Jointly, these imply that $L_m(\alpha) = \min_{j\in[k]} L_j(\alpha)$. Moreover, for all $\alpha \in [0,1]$, $q_j \leq \alpha \leq q_{j-1}$ for some $j \in [k]$. Thus if $L_m(a) = \min_{j\in[k]} L_j(\alpha)$, (64) holds.

$\square$

*Proof of Theorem 3.* Consider a $k$-ticket equilibrium defined by a win probability vector $p$ and an allocation $x$. Define the cost function

$$C(p) := \frac{1}{p}. \tag{66}$$

Define a trade reallocation $y$ of $x$. As argued in the preliminary section of Appendix , there exists a set $V$ with $\eta(V) > 0$ such that for all $v \in V$, $c \cdot y(v) \leq c \cdot x(v)$. For the remainder of the proof, we fix an arbitrary $v \in V$, define $\{u_j\}_{j=0}^k$, $\{\delta_j\}_{j=1}^k$, and $W$ for $v$ as in (51), (53), and (57), and write $x$ and $y$ for $x(v)$ and $y(v)$.

Define $B = c \cdot x$. By the definition of $W$ in (57), for any $j \in [k]$, $(\Delta, U) = (\delta_j, u_{j-1})$ satisfies (59), and therefore Lemma 7 implies

$$v \cdot y \leq \min_{j\in[k]} B\delta_j + u_{j-1}.$$

Let $m$ be an index that minimizes this bound. Then applying Lemma 8 with $\alpha = 1/B$ we see

$$q_m \leq \frac{1}{B} \leq q_{m-1}. \tag{67}$$

Furthermore, by (54) we have

$$\frac{v \cdot y}{v \cdot x} \leq \frac{B\delta_m + u_{m-1}}{u_k} = \frac{B\delta_m + u_{m-1}}{(u_k - u_{m-1}) + u_{m-1}}. \tag{68}$$

Because the righthand expression is greater than 1, it is decreasing in both $u_{m-1}$ and $u_k - u_m$, so lower bounds on these expressions imply upper bounds on trade efficiency.

To derive these bounds, we note that for $i \le j$, (62) implies

$$\frac{\delta_j}{\delta_i} = \prod_{\ell=i}^{j-1} \frac{\delta_{\ell+1}}{\delta_\ell} = \prod_{\ell=i}^{j-1}(1-q_\ell),$$

and therefore Lemma 8 implies

$$(1-q_{j-1})^{j-i} \ge \frac{\delta_j}{\delta_i} \ge (1-q_i)^{j-i}. \tag{69}$$

Rearranging gives the inequalities

$$\delta_j \ge \delta_i (1-q_i)^{j-i} \tag{70}$$

$$\delta_i \ge \frac{\delta_j}{(1-q_{j-1})^{j-i}} \tag{71}$$

for $i \le j$, which will be more convenient for our purposes.

It follows that

$$\begin{aligned} u_k - u_{m-1} = \sum_{j=m}^{k} \delta_j &\ge \delta_m \sum_{j=m}^{k} (1-q_m)^{j-m} \\ &\ge \delta_m \sum_{j=0}^{k-m} (1-1/B)^j \\ &= \delta_m B(1-(1-1/B)^{k-(m-1)}), \end{aligned} \tag{72}$$

where the first inequality is (70), the second follows from (67) and the fact that $(1-q)^\ell$ is decreasing in $q$ for any natural number $\ell$, and the final equality follows from the geometric series equation $\sum_{j=0}^{\ell} r^j = \frac{1-r^{\ell+1}}{1-r}$.

Analogously, (71) implies that

$$\begin{aligned} u_{m-1} = \sum_{j=1}^{m-1} \delta_j &\ge \delta_m \sum_{j=1}^{m-1} \frac{1}{(1-q_{m-1})^{m-j}} \\ &\ge \delta_m \sum_{j=1}^{m-1} \frac{1}{(1-1/B)^j} \\ &= \delta_m B((1-1/B)^{-(m-1)} - 1) \end{aligned} \tag{73}$$

where the second inequality follows from (67) and the fact that $\frac{1}{(1-q)^\ell}$ is increasing in $q$ for any natural number $\ell$, and the final equality follows from the geometric series equation. Substituting the lower bounds (72) and (73) into (68) and simplifying, we see that

$$\frac{v \cdot y}{v \cdot x} \leq \frac{B\delta_m + u_{m-1}}{(u_k - u_m) + u_{m-1}} \leq \frac{(1-1/B)^{-(m-1)}}{(1-1/B)^{-(m-1)} - (1-1/B)^{k-(m-1)}} = \frac{1}{1-(1-1/B)^k}.$$

The result follows from noting that $B = c \cdot x \leq \sum_{l \in [k]} c_{i_\ell} \cdot p_{i_\ell} = k$. □

### C.3.2 Proof of Theorem 4

In this section, we scale $v$ so that $w_{i_1} = 1$ without loss of generality. When $k = 2$, (54) and (57) simplify to

$$v \cdot x = 2 - q_1, \tag{74}$$

$$W(p) = \min(1, 1 - q_1 + p). \tag{75}$$

Furthermore, we have

$$x_{i_2} = p_{i_2}, \qquad x_{i_1} = (1 - p_{i_2})p_{i_1}. \tag{76}$$

The following lemmas, proved subsequently, allow us to prove Theorem 4.

**Lemma 9.** *For any $q_1 \in [0,1]$, if we define $W$ by (75) and define*

$$C(p) = \min\left(\frac{1}{p} + \frac{1}{\sqrt{2}}, \frac{\sqrt{2}}{p}\right) = \begin{cases} \frac{1}{p} + \frac{1}{\sqrt{2}} & : p \leq 2 - \sqrt{2} \\ \frac{\sqrt{2}}{p} & : p \geq 2 - \sqrt{2}, \end{cases} \tag{77}$$

$$(\Delta, U) = \begin{cases} \left(\frac{\sqrt{2}}{\sqrt{2}+q_1}, 0\right) & : q_1 \leq 2 - \sqrt{2} \\ \left(1 - q_1, \sqrt{2} - \frac{\sqrt{2}-1}{q_1}\right) & : q_1 > 2 - \sqrt{2}, \end{cases} \tag{78}$$

*then* (59) *holds.*

**Lemma 10.** *For any $q_1 \in [0,1]$, if $C$ and $(\Delta, U)$ are defined as in* (77) *and* (78)*, then*

$$\frac{v \cdot y}{v \cdot x} \leq \frac{(1+\sqrt{2})\Delta + U}{2 - q_1} \leq \frac{1+\sqrt{2}}{2}. \tag{79}$$

*Proof of Theorem 4.* The fact that any equilibrium of the 2-ticket parallel lottery is $\frac{1+\sqrt{2}}{2}$-trade efficient follows immediately from Lemma 10. We proceed to describe an example in which a trade reallocation achieves this bound.

Consider a market with two goods, and a 2-ticket equilibrium with $p_1 \leq p_2$. Define agent types

$$v^1 = (\frac{p_2}{p_1}, 1),$$
$$v^2 = (\frac{p_2}{p_1}(1 + p_1 - p_2), 1).$$

Lemma 1 implies that it is optimal for both types to place one ticket in the lottery for item 1, one ticket in the lottery for item 2, and rank item 1 above item 2. This results in the allocation

$$x(v^1) = x(v^2) = (p_1, (1 - p_1)p_2).$$

Consider an agent type measure $\eta$ such that

$$\eta(\{v^1\}) = \frac{1}{1 + \sqrt{2}}, \tag{80}$$
$$\eta(\{v^2\}) = \frac{\sqrt{2}}{1 + \sqrt{2}}, \tag{81}$$
$$\eta(\mathbb{R}^2 \setminus \{v_1, v_2\}) = 0. \tag{82}$$

This violates our assumption that the type measure is continuous, but $\eta$ can be arbitrarily well-approximated by a continuous measure.

Consider the allocation $y$ defined by

$$y(v^1) = ((1 + \sqrt{2})p_1, 0), \tag{83}$$
$$y(v^2) = (0, \frac{1 + \sqrt{2}}{\sqrt{2}}(1 - p_1)p_2), \tag{84}$$

which satisfies the trade reallocation constraint

$$x(v^1) \cdot \eta(v^1) + x(v^2) \cdot \eta(v^2) = y(v^1) \cdot \eta(v^1) + y(v^2) \cdot \eta(v^2).$$

Set $p_2 = 2 - \sqrt{2}$ and consider the limit as $p_1 \to 0$, choices which ensure that $y$ satisfies the unit demand constraint on trade reallocations. As $p_1 \to 0$, the minimum ratio by which any agent benefits under $y$ is

$$\min\left(\frac{y^1 \cdot v^1}{x^1 \cdot v^1}, \frac{y^2 \cdot v^2}{x^2 \cdot v^2}\right) = \lim_{p_1 \to 0} \min\left(\frac{(1+\sqrt{2})p_2}{(2-p_1)p_2}, \frac{\frac{1+\sqrt{2}}{\sqrt{2}}(1-p_1)p_2}{(2-p_2)p_2}\right) = \frac{1+\sqrt{2}}{2}. \qquad \square$$

*Proof of Lemma 9.* Define

$$f(p) := pC(p)\Delta + pU - W(p).$$

We wish to show that this function is non-negative. Because (75) implies that $W$ is constant on $[q_1, 1]$ and (77) implies that $pC(p)\Delta + pU$ is weakly increasing in $p$ for any non-negative $\Delta, U$, it suffices to check that $f$ is non-negative for $p \in [0, q_1]$. Furthermore, (75) implies that $W(p)$ is linear on $[0, q_1]$. Because $pC(p)\Delta + pU$ is concave, this implies that $f$ is concave on $[0, q_1]$. Thus, to show that it is non-negative on this interval, it suffices to check the endpoints. By (75) and (77),

$$f(0) = \Delta - 1 + q_1 \geq 0,$$

where the inequality follows from (78) and the fact that $1/(1 + q_1/\sqrt{2}) \geq 1 - q_1$.

Meanwhile, (75) implies that $W(q_1) = 1$. If $q_1 \leq 2 - \sqrt{2}$, then (77) implies that

$$f(q_1) = (1 + q_1/\sqrt{2})\Delta + q_1 U - 1 = 0,$$

where the second equality follows from (78). Similarly, if $q_1 > 2 - \sqrt{2}$, then (77) implies

$$f(q_1) = \sqrt{2}\Delta + q_1 U - 1 = 0. \qquad \square$$

*Proof of Lemma 10.* Lemmas 7 and 9 imply that

$$\frac{v \cdot y}{v \cdot x} \leq \frac{B\Delta + U}{2 - q_1}. \tag{85}$$

Furthermore, because

$$pC(p) = \min\left(1 + \frac{p}{\sqrt{2}}, \sqrt{2}\right), \tag{86}$$

we have

$$
\begin{aligned}
B &= x \cdot c & (87)\\
&= p_{i_2} C(p_{i_2}) + (1 - p_{i_2}) p_{i_1} C(p_{i_1}) \\
&\le p_{i_2} C(p_{i_2}) + (1 - p_{i_2}) \sqrt{2} \\
&\le 1 + p_{i_2}/\sqrt{2} + (1 - p_{i_2}) \sqrt{2} \\
&\le 1 + \sqrt{2}. & (88)
\end{aligned}
$$

The first equality above holds by definition and the second follows from (76). The first and second inequalities follow because (86) implies that $p_{i_1} C(p_{i_1}) \le \sqrt{2}$ and $p_{i_2} C(p_{i_2}) \le 1 + p_{i_2}/\sqrt{2}$. The final inequality follows because the expression on the penultimate line is decreasing in $p_{i_2}$, and thus maximized at $p_{i_2} = 0$. Combining (85) and (88) yields the first inequality in (79). We now turn to the second inequality.

If $q_1 \le 2 - \sqrt{2}$, then $(1 + q_1/\sqrt{2})(2 - q_1) \ge 2$ and therefore

$$
\frac{(1+\sqrt{2})\Delta + U}{2 - q_1} = \frac{1 + \sqrt{2}}{(1 + q_1/\sqrt{2})(2 - q_1)} \le \frac{1 + \sqrt{2}}{2}.
$$

Meanwhile, for $q_1 > 2 - \sqrt{2}$, we have

$$
\frac{(1+\sqrt{2})\Delta + U}{2 - q_1} = \frac{(1+\sqrt{2})(1 - q_1) + \sqrt{2} - \frac{\sqrt{2}-1}{q_1}}{(2 - q_1)} < \frac{1 + \sqrt{2}}{2},
$$

where the last inequality follows because the middle term evaluates to $\frac{1+\sqrt{2}}{2}$ at $q_1 = 2 - \sqrt{2}$ and the derivative of this function is

$$
-\frac{q_1^2 + 2(\sqrt{2} - 1)q_1 - 2\sqrt{2} + 2}{q_1^2 (2 - q_1)^2},
$$

which is negative for $p \in (2 - \sqrt{2}, 1]$. □

### C.4 Approximate Pareto Efficiency

*Proof of Theorem 2.* Consider a $k$-ticket equilibrium defined by a win probability vector $p$ and an allocation $x$. Define the cost function

$$C(p) := \frac{1-p}{p}, \tag{89}$$

and let $c_i = C(p_i)$.

Note that any item with $c_i > 0$ has $p_i < 1$ and therefore is fully allocated in $x$. It follows that for any feasible allocation $y$,

$$\int c \cdot y(v) d\eta \leq \int c \cdot x(v) d\eta.$$

Therefore, there exists a set of agent types $V$ with $\eta(V) > 0$ such that for all $v \in V$, $c \cdot y(v) \leq c \cdot x(v)$. Pick some $v \in V$, and define $\{w_i\}_{i\in[n]}$, $\{u_j\}_{j=0}^k$, $\{\delta_j\}_{j=1}^k$, and $W$ as in (50), (51), (53), and (57) above.

Let $i_j$ be the item on which $v$ places her $j^{\text{th}}$ ticket in her optimal strategy, and note that

$$c \cdot x \leq \sum_{l\in[k]} c_{i_\ell} \cdot p_{i_\ell} = \sum_{\ell\in[k]} 1 - p_{i_\ell} \leq k. \tag{90}$$

For any $j$, $(\Delta, U) = (\delta_j, u_j)$ satisfies (59). Because $c \cdot y \leq c \cdot x$, Lemma 7 implies

$$\frac{v \cdot y}{v \cdot x} \leq \min_{j\in[k]} \frac{\delta_j c \cdot x + u_j}{u_k}. \tag{91}$$

This immediately implies that the $k$-ticket parallel lottery is 2-Pareto efficient: taking $j = k$ in (91), we have

$$\delta_k c \cdot x \leq k\delta_k \leq u_k,$$

where the first inequality follows from (90) and the second from the fact that $\{\delta_j\}_{j=1}^k$ is a decreasing sequence as a consequence of (52) and the definition of $\{u_j\}_{j=0}^k$ in (51). Thus $v \cdot y \leq 2(v \cdot x)$.

To get a tighter bound, note that for any $m \in [k]$, we can rearrange (91) to get

$$\frac{v \cdot y}{v \cdot x} \leq \frac{(c \cdot x + 1)\delta_m + u_{m-1}}{u_k}. \tag{92}$$

Define $\{q_j\}_{j=0}^k$ in terms of $\{\delta_j\}_{j=1}^k$ as in (62), and note that Lemma 8 applies. Choose $m$ be such that $1/(c \cdot x + 1) \in [q_m, q_{m-1}]$, and note the exact correspondence between (92) and (68) in the proof of Theorem 3.

Using the same analysis as in the proof of Theorem 3, we conclude that

$$\frac{v \cdot y}{v \cdot x} \leq \frac{1}{1 - \left(1 - \frac{1}{c \cdot x + 1}\right)^k} \leq \frac{1}{1 - (1 - 1/(k+1)^k)},$$

where the final step uses (90). □

# D Proofs from Section 6

The following lemma is useful in the proofs of Propositions 4 and 5.

**Lemma 11.** *At equilibrium in the two-item setting with $\mu_1 + \mu_2 \geq 1$, almost every agent adopts one of three strategies:*

1. *Agents with $v_2 < p_1$ place $k$ tickets in the lottery for item 1 and receive $x(v) = (1-(1-p_1)^k, 0)$.*
2. *Agents with $p_1 < v_2 < 1$ place $k-1$ tickets in the lottery for item 1, place 1 ticket in the lottery for item 2, and receive $x(v) = (1 - (1-p_1)^{k-1}, (1-p_1)^{k-1})$.*
3. *Agents with $v_2 > 1$ place $k$ tickets in the lottery for item 2 and receive $x(v) = (0, 1)$.*

*Proof.* As $F(1) \geq \mu_1$, item 1 is fully allocated. As a result, $\int 1 - x_1(v)\ d\eta = 1 - \mu_1 = \mu_2$, so $p_2 = 1$ in any win probability vector $p$ corresponding to a $k$-ticket equilibrium in the two-item setting. As a result, agents with $v_2 > 1$ are guaranteed their preferred item when they adopt Strategy 3. For agents with $v_2 < 1$, every strategy that places less than $k-1$ tickets in the lottery for item 1 is dominated by Strategy 2. Finally, agents who adopt Strategy 1 have expected value $(1-(1-p_1)^{k-1})v_1 + (1-p_1)^{k-1}p_1v_1$, while agents who adopt Strategy 2 have expected value $(1-(1-p_1)^{k-1})v_1 + (1-p_1)^{k-1}p_2v_2$. Plugging in $v_1, p_2 = 1$ implies that agents with $v_2 < p_1$ adopt Strategy 1 and agents with $p_1 < v_2 < 1$ adopt Strategy 2. □

## D.1 Proof of Proposition 4

*Proof of Proposition 4.* Let $x$ be an allocation corresponding to a $k$-ticket parallel lottery in the two-item setting. Our proposed CEEI takes place in the same market, except that the two items have quantities defined by the vector

$$\tilde{\mu} := \int x(v)\, dF(v) \leq \mu. \tag{93}$$

We claim that $c := (\frac{1}{1-(1-p_1)^k}, \frac{p_1}{1-(1-p_1)^k})$ is an equilibrium price vector in this market. To see this, consider an agent $v = (1, v_2)$ who wishes to choose $(x_1, x_2)$ to maximize $x_1 + x_2 v_2$ subject to the unit demand constraint $x_1 + x_2 \leq 1$ and the budget constraint $x_1 c_1 + x_2 c_2 \leq 1$. A straightforward calculation verifies that for agents with $v_2 < p_1$, $p_1 < v_2 < 1$, and $v_2 > 1$, the allocations described in Lemma 11 solve this optimization problem and clear the market. □

## D.2 Proof of Proposition 5

*Proof of Proposition 5.* Lemma 11 implies that in any $k$-ticket equilibrium in the 2-item setting, plotting expected value in terms of $v_2$ yields a piecewise linear function (as depicted in Figure 2.) Specifically, all agents with $v_2 < p_1$ adopt Strategy 1 and have expected value $1-(1-p_1)^k$. Agents with $p_1 < v_2 < 1$ adopt Strategy 2, and expected value increases linearly to 1 on the interval $[p_1, v_2]$. Finally, agents with $v_2 > 1$ adopt Strategy 3 and have expected value $v_2$. Thus the geometry of the expected value curve implies that two such curves cross at most once.

Fix a $k$-ticket equilibrium in the two-item setting with win probability vector $p$. By Lemma 11,

$$\mu_1 = (1-(1-p_1)^k)F(p_1) + (1-(1-p_1)^{k-1})(F(1) - F(p_1)). \tag{94}$$

Rearranging this equation, we get

$$F(1) - \mu_1 = (1-p_1)^{k-1}(F(1) - p_1 F(p_1)). \tag{95}$$

By definition, the left side of this equation is a nonnegative value constant in $p$ and $k$. As a result, we have $F(1) - p_1 F(p_1) \geq 0$. Thus the right side of the equation is decreasing in $p_1$ and $k$, which implies that in the two-item setting, a smaller $k$ corresponds to a larger $p_1$ at equilibrium.

Fix $k_1 < k_2$, and let $x^{k_1}$ and $x^{k_2}$ be allocations corresponding to $k_1$ and $k_2$-ticket equilibria in the two-item setting. The win probability for item 1 is higher in the $k_1$-ticket equilibrium, so agents with small $v_2$, who adopt Strategy 1 in both equilibria, prefer $x^{k_1}$ to $x^{k_2}$. If the expected value curves corresponding to $x^{k_1}$ and $x^{k_2}$ do not cross, then every agent with $v_2 < 1$ prefers $x^{k_1}$ to $x^{k_2}$. If the expected value curves cross, then every agent with $v_2$ less than the point of indifference prefers $x^{k_1}$ to $x^{k_2}$. Thus if $v$ prefers $x^{k_1}$ to $x^{k_2}$, the same is true of any agent $w$ with $w_2 < v_2$. □